Categorical Logic

- Outline:

  1) Review of CT
  2) Propositional Calculus
  3) λ-Calculus

- Notes:

  awodey.github.io

1. Review of CT

Basic Def.s :

Category

$\mathbb{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \arrow[loop above, "1_B"] \\ & C \end{tikzcd}$$

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Functor

$$\begin{tikzcd} \mathbb{C} \arrow[dr, "F"] & \\ & \mathbb{D} \end{tikzcd}$$

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[dr, bend right, "F(g \circ f) = Fg \cdot Ff"'] & FB \arrow[r, "Fg"] \arrow[loop above, "F1_B = 1_{FB}"] & FC \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r] & E \end{tikzcd}$$

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Functor

$$\begin{tikzcd} \mathbb{C} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "\overset{\vartheta}{\Rightarrow}"] & \\ \mathbb{D} & \end{tikzcd}$$

Natural Transformation

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[dr] \arrow[dd, "\vartheta_A"'] & FB \arrow[loop above, "F1_B"] \arrow[dl, "Fg"] \arrow[dd, "\vartheta_B"] \\ & FC \arrow[dd, "\vartheta_C"'] \\ GA \arrow[r] \arrow[dr] & GB \arrow[dl, "Gg"] \\ & GC \end{tikzcd}$$

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$\vartheta_C \circ Fg = Gg \circ \vartheta_B$

Idea : " $\vartheta : F \to G$ maps the <u>structure</u> F to the <u>structure</u> G ."

Basic Examples:

- Set = Sets & functions
- Pos = Posets & monotone maps
- Top = Spaces & cts. maps
- Grp = Groups & homomorphisms
- Cat = Categories & functors

- P a poset : objects: $p, q, \ldots$
  arrows: $p \leq q$

- X a space : $\mathcal{C}(X) = \mathrm{Top}(X, \mathbb{R})$

  $f \leq g \;:=\; fx \leq gx \quad \forall\, x \in X.$

- $\mathbb{C}, \mathbb{D}$ cats : $\mathbb{D}^{\mathbb{C}} = \mathrm{Cat}(\mathbb{C}, \mathbb{D})$ functor cat

  objects: functors $F : \mathbb{C} \to \mathbb{D}$
  arrows: nat. transf. $\vartheta : F \Rightarrow G$

- $\mathbb{C}$ cat : $\hat{\mathbb{C}} := \mathrm{Set}^{\mathbb{C}^{op}}$ presheaves

$\mathbb{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \cdot f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

$\mathbb{C}^{op}$

$$\begin{tikzcd} A & B \arrow[l, "\bar{f}"'] \\ & C \arrow[u, "\bar{g}"'] \arrow[ul, "\bar{f} \cdot \bar{g}"] \end{tikzcd}$$

Representable functor

$\mathrm{Set}^{\mathbb{C}} \ni \mathbb{C}(A,-) : \mathbb{C} \longrightarrow \mathrm{Set}$

$A \in \mathbb{C}$

$B \longmapsto \mathbb{C}(A,B) = \mathrm{Hom}_{\mathbb{C}}(A,B)$

$= \{ f : A \to B \}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, bend right, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}(A,B) \arrow[d, "\mathbb{C}(A,g) := g \circ f"] \\ \mathbb{C}(A,C) \end{tikzcd}$$

Contravariant version

$\mathrm{Set}^{\mathbb{C}^{op}} \ni \mathbb{C}(-,C) : \mathbb{C}^{op} \longrightarrow \mathrm{Set}$

$C \in \mathbb{C}$

$B \longmapsto \mathbb{C}(B,C)$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ B \arrow[r, "g"'] & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}(A,C) & \\ \mathbb{C}(B,C) \arrow[u, "\mathbb{C}(f,C) := g \circ f"'] \end{tikzcd}$$

The contravariant representable is written:

$$yC := \mathbb{C}(-,C) : \mathbb{C}^{op} \longrightarrow \mathrm{Set} .$$

Def. The Yoneda embedding is the functor

$$y : \mathbb{C} \longrightarrow \mathrm{Set}^{\mathbb{C}^{op}}$$
$$C \longmapsto \mathbb{C}(-,C)$$

with action on arrows in $\mathbb{C}$ given by composition:

$$\begin{tikzcd} C \arrow[d, "h"'] \\ D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} yC \arrow[d, Rightarrow, "yh"] \\ yD \end{tikzcd},$$

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, dashed, "h\cdot g"'] & C \arrow[d, "h"] \\ & D \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}(B,C) \arrow[d] \\ \mathbb{C}(B,D) \end{tikzcd} \quad \mathbb{C}(B,h) =: (yh)_B .$$

Lemma (Yoneda) For $C \in \mathbb{C}$, $F \in \mathrm{Set}^{\mathbb{C}^{op}}$,

$$FC \cong \mathrm{Hom}(yC, F) \qquad \text{nat. in } C \,\&\, F.$$

So natural transf.s $\alpha : yC \longrightarrow F$ correspond to elements $a \in FC$.

Prop. The Yoneda embedding is full & faithful:

$$y : \mathbb{C}(C,D) \xrightarrow[\cong]{} \mathrm{Set}^{\mathbb{C}^{op}}(yC, yD)$$
$$C \xrightarrow[h]{} D \qquad yC \xrightarrow[yh]{} yD$$

Pf. $\mathrm{Hom}(yC, yD) \underset{\text{Yoneda}}{\cong} (yD)C = \mathbb{C}(C,D)$.

Adjoints

Let $\mathbb{C}, \mathbb{D}$ cats.

Def. An adjunction consists of:

- functors

$$\begin{tikzcd} \mathbb{D} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbb{C} \arrow[u, bend left, "F"] \end{tikzcd}$$

- a bijection

$$\mathbb{D}(FC, D) \cong \mathbb{C}(C, UD)$$

$$\frac{FC \longrightarrow D}{C \longrightarrow UD} \qquad \text{nat. in } C, D.$$

Examples:

1) Posets P, Q & monotone $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ s.th. $f \dashv g$:

$$p \leq gq \quad \text{iff} \quad fp \leq q.$$

E.g. take $Q = P \times P$ and $P \xrightarrow{\Delta} P \times P$.

$$p \longmapsto (p, p)$$

- What is $\Delta \dashv$?

$$(p,p) \leq (a,b) \iff p \leq a \ \& \ p \leq b$$

$$\iff p \leq a \wedge b.$$

$$\begin{tikzcd} P \times P \arrow[d, bend left, "\wedge"] \arrow[d, phantom, "\dashv"] \\ P \arrow[u, bend left, "\Delta"] \end{tikzcd}$$

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- What about $? \dashv \Delta$

$$\begin{tikzcd} P \times P \arrow[d, bend left, "\vee"] \arrow[d, phantom, "\dashv"] \\ P \arrow[u, "\Delta"] \arrow[u, phantom, "\dashv"] \arrow[u, bend right, "\wedge"] \end{tikzcd}$$

$$\frac{(a,b) \leq \overset{\Delta(p)}{(p,p)} \qquad P \times P}{\frac{a \leq p \ \& \ b \leq p \qquad P}{a \vee b \leq p \qquad P}}$$

2) Replace P by Set :

$$\begin{tikzcd} \mathrm{Set} \times \mathrm{Set} \arrow[d, bend left, "?"] \arrow[d, phantom, "\dashv"] \\ \mathrm{Set} \arrow[u, "\Delta"] \end{tikzcd}$$

So $\Delta \dashv \times$.

$\Delta(A)$

$$(A,A) \longrightarrow (B,C)$$

$$\begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & C \end{tikzcd}$$

$$A \xrightarrow[(f,g)]{} B \times C$$

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$$\frac{(A,A) \longrightarrow (B,C)}{\begin{tikzcd} & B \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & C \end{tikzcd}}$$

So $\Delta \dashv \times$.

$$\frac{A \xrightarrow[(f,g)]{} B \times C}{}$$

and of course :

$$\begin{tikzcd} \mathrm{Set} \times \mathrm{Set} \arrow[d, bend right, "+"'] \arrow[d, bend left, "\times"] \arrow[d, phantom, "\dashv \quad \dashv"] \\ \mathrm{Set} \arrow[u, "\Delta"] \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[d] \arrow[dr] & \\ B+C \arrow[r, dashed] & A \\ C \arrow[u] \arrow[ur] & \end{tikzcd}$$

<u>BTW</u> : An adjunction

$$F \dashv U$$

is always mediated by two distinguished maps :

$$\frac{FC \xrightarrow{1} FC}{C \xrightarrow[\eta]{} UFC} \quad \text{"unit"}$$

E.g. the unit of $+ \dashv \Delta$ is

$$B \xrightarrow[i_1]{} B+C \xleftarrow[i_2]{} C$$

$$\frac{UD \xrightarrow{1} UD}{FUD \xrightarrow[\varepsilon]{} D} \quad \text{"counit"}$$

3) We also have $\Delta \dashv \times$ in Pos, Cat, Grp, Top, ...

- In Pos, fix P and consider the functor

$$(-)\times P : \mathrm{Pos} \longrightarrow \mathrm{Pos} \quad .$$

- This has a <u>right adjoint</u>:

$$\begin{array}{ccc} A \times P & \longrightarrow & B \\ \hline\hline A & \longrightarrow & B^P \end{array} \qquad B^P = \mathrm{Pos}(P, B) \qquad f \leq g := f_p \leq g_p \ \forall_p$$

- We also have such right adjoints

$$(-)\times A \dashv (-)^A \quad \text{called } \underline{\text{exponentials}}$$

in Set, Cat, $\mathrm{Set}^{\mathbb{C}}$, ...

<u>Def</u>. A cat $\mathbb{C}$ is <u>cartesian closed</u> if there are right adjoints:

$$\cdot \mathbb{2} \quad \begin{tikzcd} \mathbb{1} \arrow[d, bend left, "1"] \\ \mathbb{C} \arrow[u, "!"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}\times\mathbb{C} \arrow[d, bend left, "\times"] \\ \mathbb{C} \arrow[u, "\Delta"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[d, bend left, "(-)^A"] \\ \mathbb{C} \arrow[u, "(-)\times A"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \quad .$$

$(A \in \mathbb{C})$

4) P poset <u>has $\bigvee$</u> if for all $S \subseteq P$, there's $\bigvee S \in P$ s.th.

$$\bigvee S \leq p \iff s \leq p \text{ f.all } s \in S .$$

This says that $\bigvee \dashv \downarrow$:

$$\begin{tikzcd} \mathcal{P}P \arrow[d, bend right, "\bigvee"'] \arrow[d, phantom, "\dashv"] \\ P \arrow[u, "\downarrow(-)"'] \end{tikzcd} \qquad \frac{S \subseteq \downarrow p = \{x \leq p\}}{\bigvee S \leq p}$$

<u>Prop.</u> The poset $\hat{P}$ of <u>downsets</u>,

$$\downarrow : P \to \hat{P} := \{ D \subseteq P \mid x \leq d \in D \Rightarrow x \in D \}$$

is the <u>free completion of P under $\bigvee$</u> :

$$\begin{tikzcd} \hat{P} \arrow[r, dashed, "\bar{f}"] & Q \\ P \arrow[u, "\downarrow(-)"] \arrow[ur, "f"'] & \end{tikzcd}$$

(i) $\hat{P}$ has $\bigvee$

(ii) for any Q w/ $\bigvee$ & any $f : P \to Q$ there's a unique $\bar{f}$ that preserves $\bigvee$ & $\bar{f}{\downarrow}p = fp$.

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So:

$$\bigvee\text{-}\mathrm{Pos}(\hat{P}, Q) \cong \mathrm{Pos}(P, Q) .$$

<u>Prop.</u> For any poset P, the poset $\hat{P}$ is cartesian closed & the map

$$\downarrow(-) : P \longrightarrow \hat{P}$$

preserves any cc structure in P.

Pf. $\top$ & $\wedge$ in $\hat{P}$ as in $\mathcal{P}P$.

$$\begin{aligned} A \Rightarrow B &= \bigvee \{ D \mid D \wedge A \subseteq B \} \\ &= \{ p \in P \mid \downarrow p \cap A \subseteq B \}. \end{aligned}$$

For any downset C,

$$C \leq A \Rightarrow B \quad \text{iff} \quad \downarrow c \cap A \subseteq B \quad \text{f.a. } c \in C.$$

So:

$$C \cap A = \Big(\bigcup_{c \in C} \downarrow c\Big) \cap A = \bigcup_{c \in C} (\downarrow c \cap A) \subseteq B.$$

Then:

$$\begin{aligned} \downarrow \top &= P \\ \downarrow(a \wedge b) &= \downarrow a \wedge \downarrow b \\ \downarrow(a \Rightarrow b) &= \downarrow a \Rightarrow \downarrow b \end{aligned}$$

<u>Note</u>: the map $\downarrow(-) : P \longrightarrow \hat{P}$ <u>reflects $\leq$</u>:

$$\downarrow p \leq \downarrow q = p \in \downarrow q = p \leq q.$$

In particular <u>$\downarrow$ is injective</u>.