

2. Propositional Calculus

- Consider the positive propositional Calculus:

$$PPC := P \mid T \mid \varphi \wedge \psi \mid \varphi \Rightarrow \psi$$

- The rules of inference are as usual:

$$\frac{}{\varphi \vdash T} \quad \frac{\mathcal{J} \vdash \varphi, \mathcal{J} \vdash \psi}{\mathcal{J} \vdash \varphi \wedge \psi} \quad \frac{\mathcal{J}, \varphi \vdash \psi}{\mathcal{J} \vdash \varphi \Rightarrow \psi}$$

- A Kripke model (K, \Vdash) is a poset K ,
with a relation $j \Vdash P$ s.t.h.

$$i \leq j, j \Vdash P \Rightarrow i \Vdash P \quad .$$

- Extend \Vdash to all $\varphi \in PPC$ by:

- $j \Vdash T$ always,

- $j \Vdash \varphi \wedge \psi$ iff $j \Vdash \varphi$ & $j \Vdash \psi$,

- $j \Vdash \varphi \Rightarrow \psi$ iff $i \Vdash \varphi$ implies $i \Vdash \psi$, f.a. $i \leq j$

- Let $K \Vdash \varphi$ if $j \Vdash \varphi$ f.a. $j \in K$,

- and $\Vdash \varphi$ if $K \Vdash \varphi$ f.a. (K, \Vdash) .

" φ is Kripke valid"

Prop (Kripke completeness of PPC)

$\vdash \varphi$ iff $\Vdash \varphi$.

pf. (i) Order (the formulas of) PPC by $\varphi \vdash \psi$,
& identify $\varphi = \psi$ iff $\varphi \Vdash \psi$,
Call the resulting poset \mathcal{C}_{PPC} .

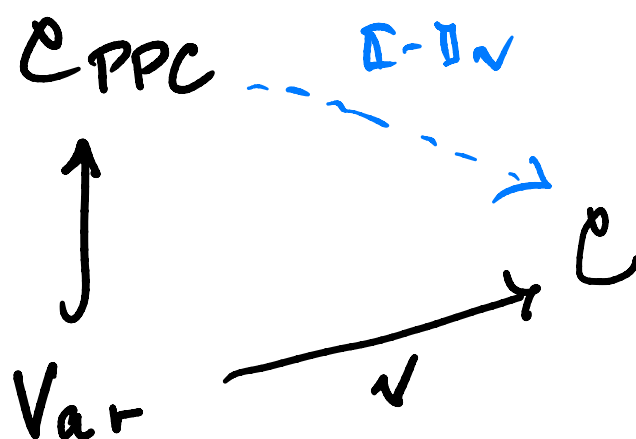
(ii) \mathcal{C}_{PPC} has:

- a terminal object T
 - products $\varphi \wedge \psi$
 - exponentials $\varphi \Rightarrow \psi$
- } a cartesian closed poset
=: CCP

(iii) C_{PPC} is the free CCP on the set

$$\{P, Q, \dots\} := \text{Var},$$

meaning:



If C is any CCP & v is any function, then there's a unique CCP map $[-]_v$ s.th.

$$[P] = vP$$

$$[T] = T$$

$$[\varphi \wedge \psi] = [\varphi] \wedge [\psi]$$

$$[\varphi \Rightarrow \psi] = [\varphi] \Rightarrow [\psi]$$

briefly:

$$\text{CCP}(C_{PPC}, C) \cong \text{Set}(\text{VAR}, C).$$

Next, we need the following ...

Lemma 1: A Kripke model (K, H) of PPC
 is the same thing as a CCP map

$$\Gamma \dashv \dashv : \mathcal{C}_{PPC} \longrightarrow \hat{K} \quad .$$

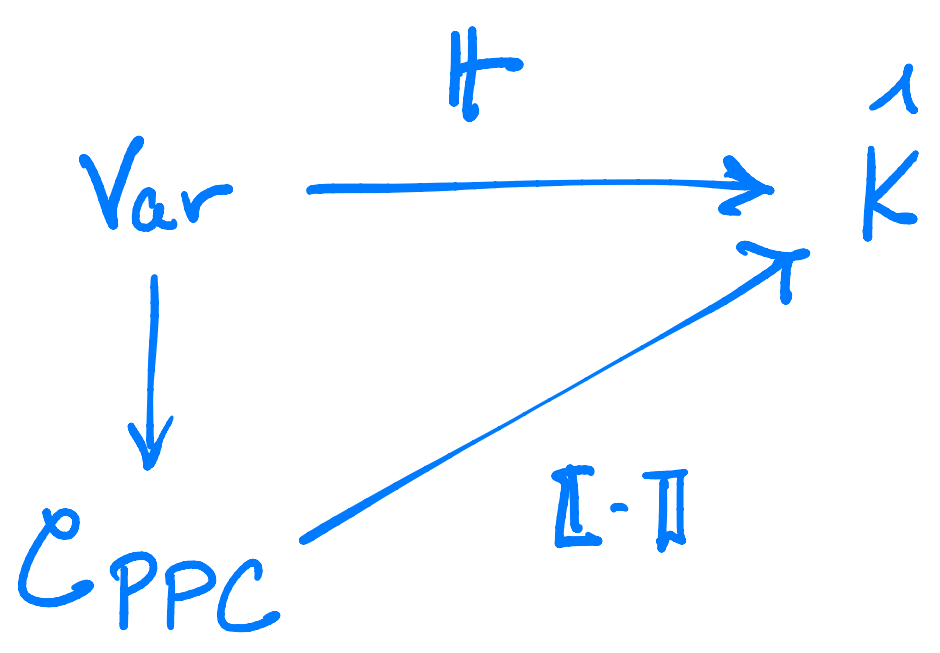
pf: There's a bijection:

$$H \subseteq K \times \text{Var} \quad \text{w/ } i \leq j \text{ \&H P} \Rightarrow i \text{ \&H P}$$

$$K \times \text{Var} \longrightarrow \mathcal{L} = (0 \leq 1) \quad \text{in Pos}$$

$$\text{Var} \longrightarrow \mathcal{L}^K = \hat{K} \quad \text{in Pos}$$

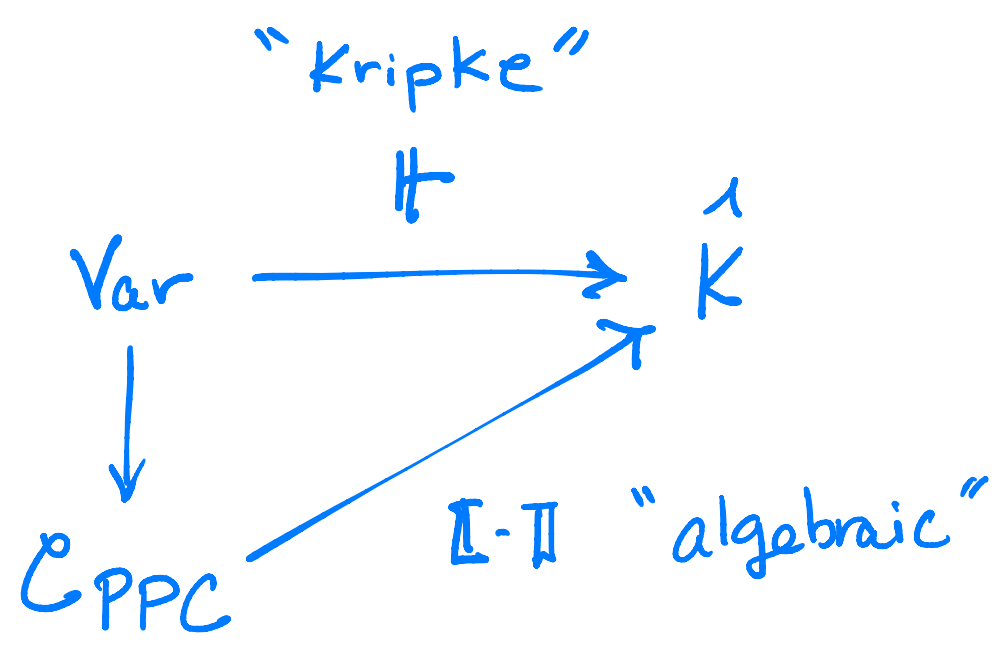
$$\Gamma \dashv \dashv : \mathcal{C}_{PPC} \longrightarrow \hat{K} \quad \text{in CCP} \quad .$$



Remark

A Kripke model (K, \Vdash) thus corresponds to an algebraic model $\llbracket - \rrbracket$ in \hat{K} via

$$j \Vdash \varphi \quad \text{iff} \quad j \in \llbracket \varphi \rrbracket .$$



The Kripke conditions correspond to algebraic ones:

$i \leq j \Vdash P \Rightarrow i \Vdash P$	$\llbracket P \rrbracket \in \hat{K}$
$j \Vdash \top$ f.a. j	$\llbracket \top \rrbracket = K$
$j \Vdash \varphi \wedge \psi$ iff $j \Vdash \varphi$ & $j \Vdash \psi$	$\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket$
$j \Vdash \varphi \Rightarrow \psi$ iff f.a. $i \leq j$, $i \Vdash \varphi$ implies $i \Vdash \psi$	$\llbracket \varphi \Rightarrow \psi \rrbracket = \llbracket \varphi \rrbracket \Rightarrow \llbracket \psi \rrbracket$

(iv) The syntactic CCP \mathcal{C}_{PPC} has a canonical algebraic model, namely

$$\downarrow : \mathcal{C}_{CCP} \rightarrow \hat{\mathcal{C}}_{CCP},$$

it corresponds to a canonical Kripke model

$$(\mathcal{C}_{PPC}, \Vdash)$$

with

$$\varphi \Vdash \psi \text{ iff } \varphi \in \Vdash \psi \text{ iff } \varphi \vdash \psi.$$

So we have:

$$(*) \quad \mathcal{C}_{PPC} \Vdash \varphi \text{ iff } \exists \delta \vdash \varphi \text{ f.a. } \delta \in \mathcal{C}_{PPC} \\ \text{iff } \vdash \varphi.$$

(v) Thus we have:

$$\Vdash \varphi = K \Vdash \varphi \text{ f.a. } (K, \Vdash)$$

$$\Rightarrow \mathcal{C}_{PPC} \Vdash \varphi$$

$$\stackrel{*}{\Leftrightarrow} \vdash \varphi. \text{ Completeness } \checkmark$$

Conversely:

$$\vdash \varphi \stackrel{*}{\Leftrightarrow} \mathcal{C}_{PPC} \Vdash \varphi \text{ Soundness}$$

$$\Rightarrow K \Vdash \varphi \text{ f.a. } (K, \Vdash)$$

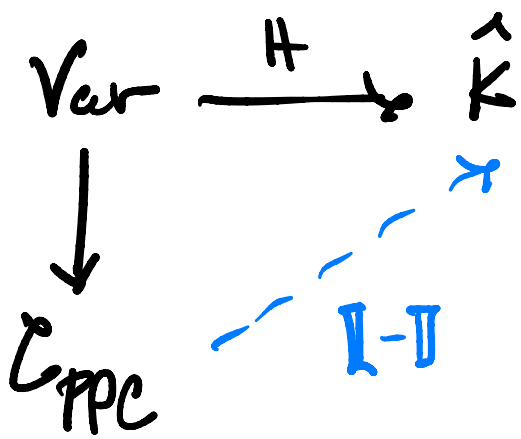
by the following ...

Lemma 2: Given $\varphi \in \text{PPC}$,

$\mathcal{C}_{\text{PPC}} \Vdash \varphi$ implies $K \Vdash \varphi$ f.a. (K, H) .

pf. $\mathcal{C}_{\text{PPC}} \Vdash \varphi \Rightarrow T \Vdash \varphi$
 $\Rightarrow T \vdash \varphi$
 $\Rightarrow T \leq \varphi$ in \mathcal{C}_{PPS} .

Given any (K, H) we get a PPC map $\llbracket - \rrbracket$:



Thus:

$$\llbracket T \rrbracket \leq \llbracket \varphi \rrbracket$$

So:

$$K \Vdash \varphi.$$

Kripke completeness for PPC ✓

Next we can extend this result to:

(1) IPC = PPC w/ \perp, \vee (Kripke)

(2) Topological semantics (Tarski)

(3) A translation (Gödel)

$$\text{IPC} \longleftrightarrow \text{CPC w/ } \#$$

(Discuss each one briefly)

1. Extension from PPC to IPC

(9)

Def. A Heyting algebra is a CCP with all finite joins: $\perp, p \vee q, \dots$

Equivalently, a bdd. lattice w/ $a \Rightarrow b$.

Examples

i. any Boolean algebra \mathcal{B} is a HA,

$$p \Rightarrow q := \neg p \vee q$$

ii. for any topological space (X, \mathcal{O}_X) , the open sets form a HA,

$$U \Rightarrow V = \bigcup_{W \cap U \subseteq V} W$$

iii. any V -complete distributive lattice, e.g. \hat{P} f. any poset P .

iv. any complete linear order,

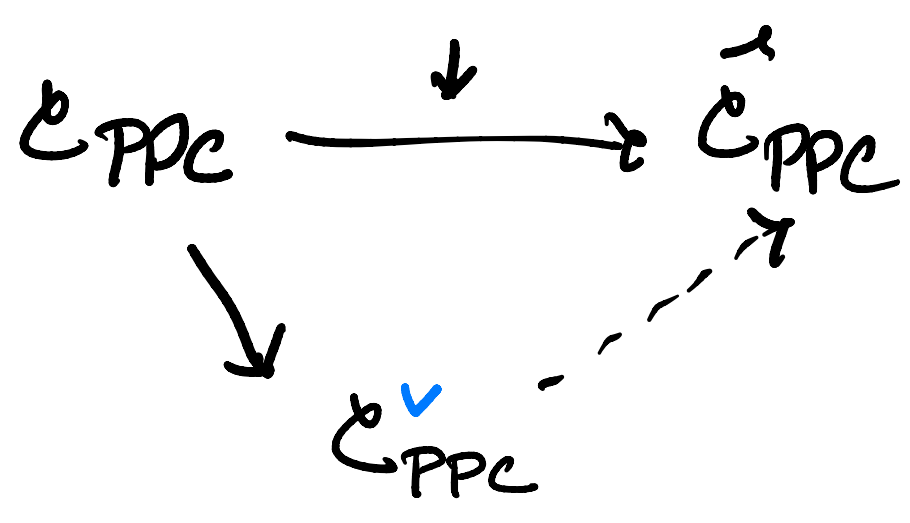
e.g. $[0, 1] \subseteq \mathbb{R}$. "fuzzy logic"

By (iii) the canonical model of PPC

$$\downarrow : \mathcal{C}_{PPC} \longrightarrow \hat{\mathcal{C}}_{PPC}$$

is there for valued in a HA !

Since being a HA is algebraic, there's a unique extension to the V-completion, from CCPs to HAs :

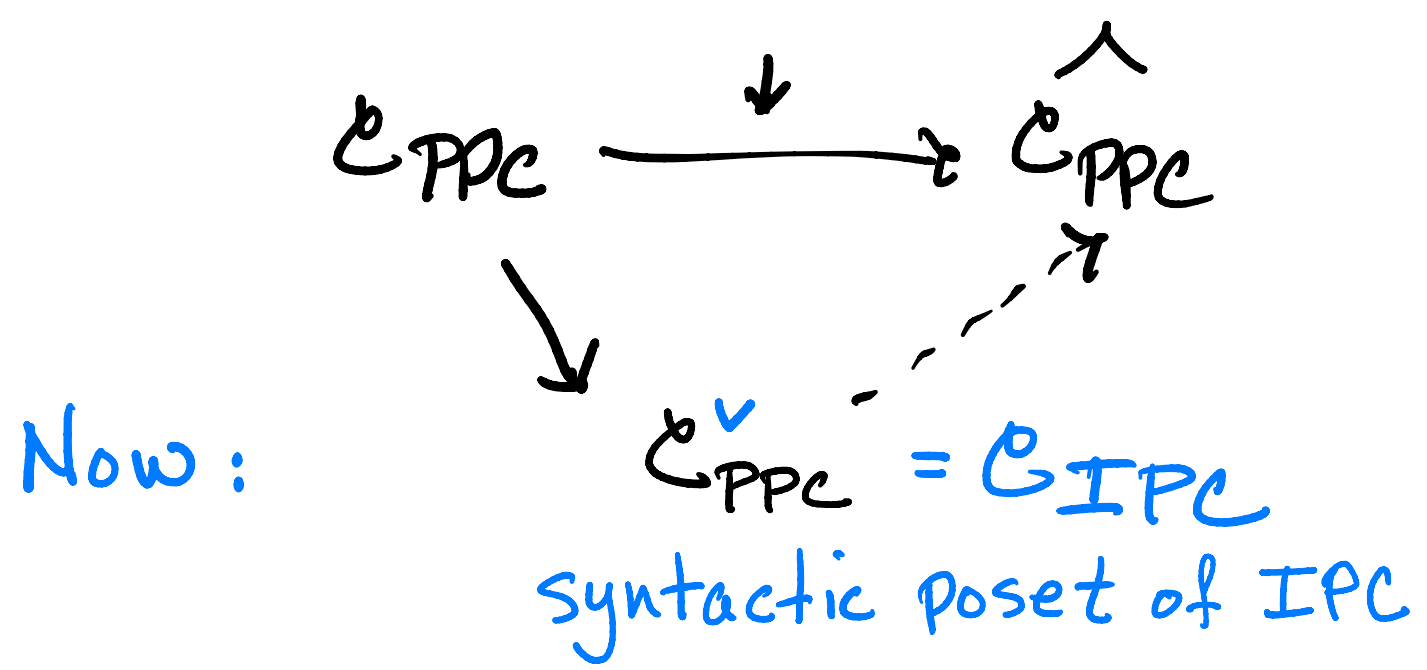


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And for φ, ψ in IPC, we again have:

$$\varphi \vdash_{IPC} \psi \iff \downarrow \varphi \leq \downarrow \psi \text{ in } \hat{\mathcal{C}}_{IPC}$$

But unfortunately the CCP embedding

$$\downarrow : \mathcal{C}_{IPC} \longrightarrow \hat{\mathcal{C}}_{IPC}$$

is not Heyting (it does not preserve \perp, \vee).

Instead, we use the following:

Thm (Joyal) For H any $H\Lambda$ the map

$$j : H \longrightarrow \widehat{H^*},$$

where $H^* = \text{Prime}(H)$

and $j(h) = \{p \mid h \in p\}$

is both Heyting and conservative.

$$jx \leq jy \Rightarrow x \leq y$$

Generalizes Stone Representation Theorem from BAs to HAs.

Pf: Uses Birkhoff's prime ideal theorem.

Now we proceed as for PPC $\xleftrightarrow{\downarrow} \widehat{PPC}$,
but using IPC $\xleftrightarrow{j} \widehat{IPC}^*$ instead:

Thm. (Completeness of IPC, Kripke 1965)

Let IPC = PPC extended by:

$$\frac{\bullet}{\perp \vdash \varphi} \qquad \frac{\varphi \vdash \mathcal{D} \quad \psi \vdash \mathcal{D}}{\varphi \vee \psi \vdash \mathcal{D}}$$

Then:

$$\vdash \varphi \quad \text{iff} \quad K \Vdash \varphi \quad \text{f.a. Kripke models } (K, \Vdash)$$

Here K is a poset and again

$$\Vdash \subseteq \widehat{K} \times \text{Var},$$

extended to IPC \supseteq Var by:

- $j \Vdash \top$ f.a. j
- $j \not\Vdash \perp$ f.a. j
- $j \Vdash \varphi \wedge \psi$ iff $j \Vdash \varphi$ & $j \Vdash \psi$
- $j \Vdash \varphi \vee \psi$ iff $j \Vdash \varphi$ or $j \Vdash \psi$
- $j \Vdash \varphi \Rightarrow \psi$ iff $i \Vdash \varphi$ implies $i \Vdash \psi$ for $i \leq j$

2. Extension to Topological Semantics

(13)

Def. A topological model of IPC consists of a space X and a HA homomorphism

$$\Vdash : \mathcal{C}_{IPC} \longrightarrow \mathcal{O}X .$$

Note: this can be unwound in the expected way:

$$\Vdash \top = X$$

$$\Vdash \varphi \wedge \psi = \Vdash \varphi \cap \Vdash \psi$$

etc.

Say a formula φ is (topologically) valid if

$$\Vdash \varphi = X \quad \text{f.a. } (X, \Vdash).$$

Thm. (Tarski 1938)

A formula φ is topologically valid iff it is provable in IPC.

Pf. This follows directly from Joyal's thm:

Take the space of prime ideals in \mathcal{C}_{IPC} ,

$$\text{Spec}(IPC) = \mathcal{C}_{IPC}^*$$

topologized with the "Zariski" basic opens:

$$B_q = \{ \mathfrak{p} \mid q \in \mathfrak{p} \}.$$

3. Extension to Modal Logic

We have an embedding of HAS,

$$\mathbb{I} \dashv \dashv : \mathcal{L}_{IPC} \longleftrightarrow \mathcal{O}Spec(IPC)$$

For any space X , the interior operation

$$o : \mathcal{P}X \longrightarrow \mathcal{P}X$$

provides a (topological) model of the modal logic IPC :

$$\frac{\varphi \vdash \psi}{\Box \varphi \vdash \Box \psi} \quad \cdot \quad \Box \varphi \vdash \varphi \quad \cdot \quad \Box \varphi \vdash \Box \Box \varphi$$
$$\Box \varphi \vdash \Box \psi \quad \cdot \quad \top \vdash \Box \top \quad \cdot \quad \Box \varphi \wedge \Box \psi \vdash \Box (\varphi \wedge \psi)$$

Thm. (McKinsey - Tarski 1944)

CPC^{\Box} is complete with respect to topological models.

Remark : One can also use this to prove the completeness of Gödel's translation

$$IPC \longrightarrow \Box CPC$$