2. Propositional Calculus

- Consider the positive propositional calculus:

$$\mathrm{PPC} := p \mid \top \mid \varphi \wedge \psi \mid \varphi \Rightarrow \psi$$

- The rules of inference are as usual:

$$\frac{\cdot}{\varphi \vdash \top} \qquad \frac{\vartheta \vdash \varphi \,,\; \vartheta \vdash \psi}{\vartheta \vdash \varphi \wedge \psi} \qquad \frac{\vartheta, \varphi \vdash \psi}{\vartheta \vdash \varphi \Rightarrow \psi}$$

- A Kripke model $(K, \Vdash)$ is a poset K, with a relation $j \Vdash p$ s.th.

$$i \leq j \,,\; j \Vdash p \;\Rightarrow\; i \Vdash p \,.$$

- Extend $\Vdash$ to all $\varphi \in \mathrm{PPC}$ by:
  - $j \Vdash \top$ always,
  - $j \Vdash \varphi \wedge \psi$ iff $j \Vdash \varphi$ & $j \Vdash \psi$,
  - $j \Vdash \varphi \Rightarrow \psi$ iff $i \Vdash \varphi$ implies $i \Vdash \psi$, f.a. $i \leq j$

- Let $K \Vdash \varphi$ if $j \Vdash \varphi$ f.a. $j \in K$, and $\Vdash \varphi$ if $K \Vdash \varphi$ f.a. $(K, \Vdash)$.

"φ is Kripke valid"

Prop (Kripke completeness of PPC)

$$\vdash \varphi \quad \text{iff} \quad \Vdash \varphi .$$

pf. (i) Order (the fmlas of) PPC by $\varphi \vdash \psi$,
& identify $\varphi = \psi$ iff $\varphi \dashv\vdash \psi$.
Call the resulting poset $\mathcal{C}_{PPC}$.

(ii) $\mathcal{C}_{PPC}$ has:

- a terminal object $\top$
- products $\varphi \wedge \psi$
- exponentials $\varphi \Rightarrow \psi$

a cartesian closed poset =: CCP

(iii) $\mathcal{C}_{PPC}$ is the free CCP on the set

$$\{p, q, \dots\} := \mathrm{Var},$$

meaning:

$$\begin{tikzcd} \mathcal{C}_{PPC} \arrow[dr, dashed, "{[\![-]\!]_v}"] & \\ \mathrm{Var} \arrow[u, hook] \arrow[r, "v"'] & \mathcal{C} \end{tikzcd}$$

If $\mathcal{C}$ is any CCP & v is any function, then there's a unique CCP map $[\![-]\!]_v$ s.th.

$$\begin{aligned} [\![p]\!] &= vp \\ [\![\top]\!] &= \top \\ [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \wedge [\![\psi]\!] \\ [\![\varphi \Rightarrow \psi]\!] &= [\![\varphi]\!] \Rightarrow [\![\psi]\!], \end{aligned}$$

briefly:

$$\mathrm{CCP}(\mathcal{C}_{PPC}, \mathcal{C}) \cong \mathrm{Set}(\mathrm{VAR}, \mathcal{C}).$$

Next, we need the following ...

Lemma 1: A Kripke model $(K, \Vdash)$ of PPC is the same thing as a CCP map

$$[\![-]\!] : \mathcal{C}_{PPC} \longrightarrow \hat{K}.$$

Pf: There's a bijection:

$$\Vdash \; \subseteq \; K \times Var \qquad \text{w/ } i \leq j \Vdash p \Rightarrow i \Vdash p$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$K \times Var \longrightarrow 2 = (0 \leq 1) \qquad \text{in Pos}$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$Var \longrightarrow 2^K = \hat{K} \qquad \text{in Pos}$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$[\![-]\!] : \mathcal{C}_{PPC} \longrightarrow \hat{K} \qquad \text{in CCP}.$$

$$\begin{tikzcd}
Var \arrow[r, "\Vdash"] \arrow[d] & \hat{K} \\
\mathcal{C}_{PPC} \arrow[ur, "{[\![-]\!]}"'] &
\end{tikzcd}$$

Remark

A Kripke model $(K, \Vdash)$ thus corresponds to an algebraic model $[\![-]\!]$ in $\hat{K}$ via

$$j \Vdash \varphi \quad \text{iff} \quad j \in [\![\varphi]\!] \quad .$$

$$\begin{tikzcd}
\mathrm{Var} \arrow[rr, "\Vdash", "\text{``Kripke''}"'] \arrow[d] & & \hat{K} \\
\mathcal{C}_{PPC} \arrow[urr, "{[\![-]\!]} \ \text{``algebraic''}"'] & &
\end{tikzcd}$$

The Kripke conditions correspond to algebraic ones:

$i \leq j \Vdash p \Rightarrow i \Vdash p$	$[\![p]\!] \in \hat{K}$
$j \Vdash \top$ f.a. j	$[\![\top]\!] = K$
$j \Vdash \varphi \wedge \psi$ iff $j \Vdash \varphi$ & $j \Vdash \psi$	$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$
$j \Vdash \varphi \Rightarrow \psi$ iff f.a. $i \leq j$, $i \Vdash \varphi$ implies $i \Vdash \psi$	$[\![\varphi \Rightarrow \psi]\!] = [\![\varphi]\!] \Rightarrow [\![\psi]\!]$.

(iv) The syntactic CCP $\mathcal{C}_{PPC}$ has a canonical algebraic model, namely

$$\downarrow : \mathcal{C}_{CCP} \longrightarrow \hat{\mathcal{C}}_{CCP} ,$$

it corresponds to a canonical Kripke model

$$(\mathcal{C}_{PPC}, \Vdash) ;$$

with

$$\varphi \Vdash \psi \quad \text{iff} \quad \varphi \in [\![\psi]\!] \quad \text{iff} \quad \varphi \vdash \psi .$$

So we have:

$$(*) \qquad \mathcal{C}_{PPC} \Vdash \varphi \quad \text{iff} \quad \gamma \vdash \varphi \quad \text{f.a. } \gamma \in \mathcal{C}_{PPC}$$
$$\text{iff} \quad \vdash \varphi .$$

(v) Thus we have:

$$\Vdash \varphi \quad = \quad K \Vdash \varphi \quad \text{f.a. } (K, \Vdash)$$
$$\Rightarrow \quad \mathcal{C}_{PPC} \Vdash \varphi$$
$$\overset{\Leftrightarrow}{*} \quad \vdash \varphi . \qquad \text{Completeness} \checkmark$$

Conversely:

$$\vdash \varphi \quad \overset{\Leftrightarrow}{*} \quad \mathcal{C}_{PPC} \Vdash \varphi \qquad \text{Soundness}$$
$$\Rightarrow \quad K \Vdash \varphi \quad \text{f.a. } (K, \Vdash)$$

by the following ...

Lemma 2: Given $\varphi \in PPC$,

$\mathcal{C}_{PPC} \Vdash \varphi$ implies $K \Vdash \varphi$ f.a. $(K, \Vdash)$.

pf. $\mathcal{C}_{PPC} \Vdash \varphi \Rightarrow T \Vdash \varphi$

$\Rightarrow T \vdash \varphi$

$\Rightarrow T \leq \varphi$ in $\mathcal{C}_{PPS}$.

Given any $(K, \Vdash)$ we get a PPC map $[\![-]\!]$:

$$\begin{tikzcd} Var \arrow[r, "\Vdash"] \arrow[d] & \hat{K} \\ \mathcal{C}_{PPC} \arrow[ur, dashed, "{[\![-]\!]}"'] & \end{tikzcd}$$

Thus:

$[\![T]\!] \leq [\![\varphi]\!]$

So:

$K \Vdash \varphi$.

Kripke completeness for PPC ✓

Next we can extend this result to:

(1) IPC = PPC w/ $\perp, \vee$ (Kripke)

(2) Topological semantics (Tarski)

(3) A translation (Gödel)

$$\mathrm{IPC} \hookrightarrow \mathrm{CPC} \text{ w/ } \square$$

(Discuss each one briefly)

1. Extension from PPC to IPC

<u>Def.</u> A <u>Heyting algebra</u> is a CCP with all finite joins: $\bot$, $p \vee q$, ...

Equivalently, a bdd. lattice w/ $a \Rightarrow b$.

<u>Examples</u>

i. any Boolean algebra $\mathcal{B}$ is a HA,

$$p \Rightarrow q := \neg p \vee q$$

ii. for any topological space $(X, \mathcal{O}_X)$, the open sets form a HA,

$$U \Rightarrow V = \bigcup_{W \cap U \subseteq V} W$$

iii. any $\bigvee$-complete distributive lattice, e.g. $\hat{P}$ f. any poset P.

iv. any <u>complete</u> linear order, e.g. $[0,1] \subseteq \mathbb{R}$. "fuzzy logic"

By (iii) the canonical model of PPC

$$\downarrow : \mathcal{C}_{PPC} \longrightarrow \hat{\mathcal{C}}_{PPC}$$

is therefor valued in a HA !

Since being a HA is algebraic, there's a unique extension to the $\vee$-Completion, from CCPs to HAs:

$$\begin{tikzcd}
\mathcal{C}_{PPC} \arrow[rr, "\downarrow"] \arrow[dr] & & \hat{\mathcal{C}}_{PPC} \\
& \mathcal{C}^{\vee}_{PPC} \arrow[ur, dashed] &
\end{tikzcd}$$

By (iii) the canonical model of PPC

$$\downarrow : \mathcal{C}_{PPC} \longrightarrow \hat{\mathcal{C}}_{PPC}$$

is therefor valued in a HA !

Since being a HA is algebraic, there's a unique extension to the $\vee$-completion, from CCPs to HAs:

$$\begin{tikzcd} \mathcal{C}_{PPC} \arrow[rr, "\downarrow"] \arrow[dr] & & \hat{\mathcal{C}}_{PPC} \\ & \mathcal{C}^{\vee}_{PPC} \arrow[ur, dashed] & \end{tikzcd}$$

Now: $\qquad \mathcal{C}^{\vee}_{PPC} = \mathcal{C}_{IPC}$

syntactic poset of IPC

And for φ, ψ in IPC, we again have:

$$\varphi \vdash_{IPC} \psi \iff \downarrow\varphi \leq \downarrow\psi \text{ in } \hat{\mathcal{C}}_{IPC}.$$

But unfortunately the CCP embedding

$$\downarrow : \mathcal{C}_{IPC} \longrightarrow \hat{\mathcal{C}}_{IPC}$$

is not Heyting (it does not preserve $\bot, \vee$).

Instead, we use the following:

Thm (Joyal) For H any HA the map

$$j : H \longrightarrow \widehat{H^*},$$

where $$H^* = \mathrm{Prime}(H)$$

and $$j(h) = \{ p \mid h \in p \}$$

is both Heyting and conservative.

$jx \leq jy \Rightarrow x \leq y$

Generalizes Stone Representation Theorem from BAs to HAs.

Pf: Uses Birkhoff's prime ideal theorem.

Now we proceed as for $PPC \overset{\downarrow}{\hookrightarrow} \widehat{PPC}$,
but using $IPC \overset{j}{\hookrightarrow} \widehat{IPC}^*$ instead:

<u>Thm.</u> (Completeness of IPC, Kripke 1965)

Let IPC = PPC extended by:

$$\frac{\cdot}{\bot \vdash \varphi} \qquad \frac{\varphi \vdash \vartheta \qquad \psi \vdash \vartheta}{\varphi \vee \psi \vdash \vartheta}$$

Then:

$$\vdash \varphi \quad \text{iff} \quad K \Vdash \varphi \ \text{f.a. Kripke models } (K, \Vdash) \ .$$

Here K is a poset and again

$$\Vdash \ \subseteq \ \hat{K} \times Var \ ,$$

extended to $IPC \supseteq Var$ by:

- $j \Vdash \top$ f.a. j
- $j \not\Vdash \bot$ f.a. j
- $j \Vdash \varphi \wedge \psi$ iff $j \Vdash \varphi$ & $j \Vdash \psi$
- $j \Vdash \varphi \vee \psi$ iff $j \Vdash \varphi$ or $j \Vdash \psi$
- $j \Vdash \varphi \Rightarrow \psi$ iff $i \Vdash \varphi$ implies $i \Vdash \psi$ for $i \leq j$.

2. Extension to Topological Semantics

Def. A topological model of IPC consists of a space X and a HA homomorphism

$$[\![-]\!] : \mathcal{C}_{IPC} \longrightarrow \mathcal{O}X .$$

Note: this can be unwound in the expected way:

$$[\![\top]\!] = X$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$

etc.

Say a fmla φ is (topologically) valid if

$$[\![\varphi]\!] = X \qquad \text{f.a. } (X, [\![-]\!]).$$

Thm. (Tarski 1938)

A fmla φ is topologically valid iff it is provable in IPC.

Pf. This follows directly from Joyal's thm: Take the space of prime ideals in $\mathcal{C}_{IPC}$,

$$\mathrm{Spec}(IPC) = \mathcal{C}^*_{IPC} ,$$

topologized with the "Zariski" basic opens:

$$B_q = \{ p \mid q \in p \}.$$

3. Extension to Modal Logic

We have an embedding of HAs,

$$\Pi\text{-}\Sigma : \mathcal{C}_{IPC} \hookrightarrow \mathcal{O}\,\mathrm{Spec}(IPC)$$

For any space X, the interior operation

$$\circ : \mathcal{P}X \longrightarrow \mathcal{P}X$$

provides a (topological) model of the modal logic $\square CPC$:

$$\frac{\varphi \vdash \psi}{\square\varphi \vdash \square\psi} \quad \bullet\ \square\varphi \vdash \varphi \quad \bullet\ \square\varphi \vdash \square\square\varphi$$

$$\bullet\ \top \vdash \square\top \quad \bullet\ \square\varphi \wedge \square\psi \vdash \square(\varphi\wedge\psi)$$

Thm. (McKinsy-Tarski 1944)

$CPC^{\square}$ is complete with respect to topological models.

Remark: One can also use this to prove the completeness of Gödel's translation

$$IPC \longrightarrow \square CPC.$$