3. λ-Calculus

For propositional logic PL we now have 3 different kinds of models (say):

- Kripke : $PL \longrightarrow \hat{K} = 2^K$
- Topological : $PL \longrightarrow \mathcal{O}X$
- Algebraic : $PL \longrightarrow \mathcal{P}X \circlearrowleft \Box$

One also has the same for first-order (predicate) logic FOL, of all 3 kinds: coherent, intuitionistic, and classical.

For example, there are both Kripke & topological semantics for IFOL and for $\Box$CFOL, as well as algebraic semantics for the latter.*

* References in the notes!

<u>Here</u> we'll generalize in another way: from <u>Propositions to Types</u>.

$1, \times, \to$
$(0, +)$

$$\begin{tikzcd}
\mathrm{STT} \arrow[d, dash] & \\
\mathrm{PL} \arrow[r, dash] & \mathrm{FOL}
\end{tikzcd}$$

$\top, \wedge, \Rightarrow$
$(\bot, \vee)$

$\exists, \forall$
$\top, \wedge, \Rightarrow$
$(\bot, \vee)$

<u>Here</u> we'll generalize in another way: from <u>Propositions to Types</u>.

$$\begin{tikzcd}
1, \times, \to \\ (0, +) & & & 1, \Sigma, \Pi \\ (0, 2) \\
& \mathrm{STT} \arrow[r, dash] \arrow[d, dash] & \mathrm{DTT} \arrow[d, dash] & \\
& \mathrm{PL} \arrow[r, dash] & \mathrm{FOL} & \exists, \forall \\ \top, \wedge, \Rightarrow \\ (\bot, \vee) \\
\top, \wedge, \Rightarrow \\ (\bot, \vee) & & &
\end{tikzcd}$$

For STT we also have all 3 kinds of <u>semantics</u>:

- Kripke: $\mathbf{Set}^{\mathbb{C}^{op}}$ presheaves
- Topological: $\mathrm{Sh}(X)$ sheaves
- Algebraic: $\mathbf{Set}/_X \circlearrowleft \sharp$ coalgebras

But first let's consider the idea of

<u>Propositions as Types</u>

from a categorical point of view.

The Curry-Howard-Scott-Lawvere-Tait-Martin-Löf Correspondence

As a cat, the poset PL of propositional fmlas has as arrows mere entailments:

$$\varphi \leq \psi \quad \text{iff} \quad \varphi \vdash \psi$$

But this discards some information, namely how $\varphi \vdash \psi$ was established,

$$\alpha \; \begin{array}{c} \varphi \\ \hline \vdots \\ \hline \psi \end{array} \quad .$$

And there may be many different proofs

$$\alpha \; \begin{array}{c} \varphi \\ \hline \vdots \\ \hline \psi \end{array} \qquad \beta \; \begin{array}{c} \varphi \\ \hline \vdots \\ \hline \psi \end{array} \qquad \gamma \; \begin{array}{c} \varphi \\ \hline \vdots \\ \hline \psi \end{array}$$

of the same entailment $\varphi \vdash \psi$.

In addition to the poset PL, there is also the evident category of proofs:

$$\left(\alpha\;\begin{array}{c} \varphi \\ \hline \vdots \\ \hline \psi \end{array} \;,\; \beta\;\begin{array}{c} \psi \\ \hline \vdots \\ \hline \vartheta \end{array} \right) \longmapsto \begin{array}{c} \varphi \\ \hline \vdots \\ \psi \\ \vdots \\ \hline \vartheta \end{array} \;\; \beta \circ \alpha$$

objects: fmlas $\varphi, \psi, \ldots$

arrows: proofs $\alpha : \varphi \vdash \psi$

Now we can make use of the following

Fact: Fmlas & proofs of PL w/ $1, \wedge, \Rightarrow$ are described exactly by types & terms of simply-typed λ-calculus:

$$\text{Proofs} \cong \lambda\text{-Terms} \, .$$

E.g. Consider the entailment:

$$p \wedge q \vdash (p \Rightarrow q) \Rightarrow q$$

and the two proofs:

$$\dfrac{\dfrac{\dfrac{p \wedge q}{p} \quad (p \Rightarrow q)_1}{q}}{(p \Rightarrow q) \Rightarrow q}\,1 \qquad\qquad \dfrac{\dfrac{p \wedge q}{q} \quad (p \Rightarrow q)_1}{(p \Rightarrow q) \Rightarrow q}\,1$$

we can record the difference by annotating them with proof-terms:

$$\dfrac{\dfrac{\dfrac{x : p \wedge q}{\pi_1 x : p} \quad y : p \Rightarrow q}{y(\pi_1 x) : q}}{\lambda y.\, y(\pi_1 x) : (p \Rightarrow q) \Rightarrow q} \qquad\qquad \dfrac{\dfrac{x : p \wedge q}{\pi_2 x : q} \quad y : p \Rightarrow q}{\lambda y.\, \pi_2 x : (p \Rightarrow q) \Rightarrow q}$$

These determine 2 different arrows

$$\begin{tikzcd} p \wedge q \arrow[r, bend left, "\lambda y.\, y(\pi_1 x)"] \arrow[r, bend right, "\lambda y.\, \pi_2 x"'] & (p \Rightarrow q) \Rightarrow q \end{tikzcd}$$

in the category of proofs $\mathcal{C}_{STT}$.

<u>Def.</u> For any Cat $\mathbb{C}$, let $|\mathbb{C}|$ be the <u>poset reflection</u> of $\mathbb{C}$, with

objects: $A, B, \ldots$ as in $\mathbb{C}$

arrows: $A \leq B$ iff $\exists f : A \to B \in \mathbb{C}$.

There's an evident functor

$$\mathbb{C} \longrightarrow |\mathbb{C}|$$

which is <u>universal</u> among functors into <u>posets</u>:

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & P \\ {|\mathbb{C}|} \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

Cat Pos

Put differently, the poset reflection functor $|\cdot| : \mathbf{Cat} \to \mathbf{Pos}$ is <u>left adjoint</u> to the <u>inclusion</u>:

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend right, "|\cdot|"'] \arrow[r, phantom, "\top"] & \mathbf{Pos} \arrow[l, bend right] \end{tikzcd} .$$

<u>Prop. (G-H)</u>

$$|\mathcal{C}_{STT}| = \mathcal{C}_{PL} .$$

Digression on HoTT

The 2 levels of Propositions as Types are thus related to ones in CT:

$$\begin{array}{ccc} \text{Type} & \sim & \text{Cat} \\ | & & | \\ \text{Prop} & \sim & \text{Pos} \end{array}$$

The idea of proof-relevance also has an analogue in CT, called:

Categorification:

A higher categorical structure with a lower categorical one as its truncated form:

Cat $\quad P \xrightarrow{i_1} P+Q \xleftarrow{i_2} Q$

Pos
$$\begin{array}{ccccc} & & p \vee q & & \\ & / & & \backslash & \\ p & & & & q \end{array}$$

Categorification occurs also in <u>higher dimensions</u>:

∞Cat

$\vdots$

3-Cat

|

2-Cat $\qquad P \oplus Q \simeq Q \oplus P$

|

Cat $\qquad P + Q \cong Q + P$

|

Pos $\qquad p \vee q = q \vee p$

In HoTT we have learned that this also happens in TT:

∞type

$\vdots$

Type $\quad$ 2 type

| $\quad$ 1 type

Prop $\quad$ 0 type

So a better slogan might be:

Propositions as Homotopy Types!

Def.s In STT with $1, \times, \to$:

(1) A **λ-theory** $\mathbb{T}$ consists of

- Basic types $B_1, B_2, \dots$
- Basic terms $b_1 : X_1,\ b_2 : X_2, \dots$
- Equations $s_1 = t_1 : E_1,\ s_2 = t_2 : E_2, \dots$

(2) $\mathbb{T} \vdash t : T$ means $\cdot \vdash t : T$ in $\mathbb{T}$.

$\mathbb{T} \vdash s = t : E$ means $\cdot \vdash s = t : E$ in $\mathbb{T}$.

Examples

(1) **Groups** : G, $u : G$

$m : G \times G \to G$

$i : G \to G$

$m(u, g) = g : G$

...

(2) **Reflexive Domains** : D, $s : (D \to D) \to D$

$r : D \to (D \to D)$

$r \circ s = 1_D,\ s \circ r = 1_{D \to D}$

(3) **HOL** : Ω, $\top, \bot : \Omega$

(HA equations) $\wedge, \vee, \Rightarrow : \Omega \longrightarrow \Omega$

(Lambek-Scott equations) $\exists_X, \forall_X : (X \to \Omega) \longrightarrow \Omega$ f.a. X

Def.s In a CCC $\mathcal{C}$:

(3) A **$\mathbb{T}$-model** $\mathcal{M}$ consists of

- objects $[\![B_1]\!], [\![B_2]\!], \ldots$
- arrows $[\![b_1]\!] : 1 \to [\![X_1]\!], \ldots$

  where $[\![X \times Y]\!] = [\![X]\!] \times [\![Y]\!]$

  $[\![X \to Y]\!] = [\![Y]\!]^{[\![X]\!]}$
- s.th. $[\![s_1]\!] = [\![t_1]\!] : 1 \longrightarrow [\![E_1]\!]$

  $\ldots$

(4) A model $\mathcal{M}$ **inhabits** a type T :

$$\mathcal{M} \models T \quad := \quad \exists\, 1 \longrightarrow [\![T]\!] \text{ in } \mathcal{C} \,,$$

A model $\mathcal{M}$ **satisfies** an equation:

$$\mathcal{M} \models s = t$$

$$:= \quad [\![s]\!] = [\![t]\!] : 1 \longrightarrow [\![E]\!] \,.$$

Examples

(1) If $\mathbb{T} = \text{Groups}$, $\mathcal{C} = \text{Set}$, then a $\mathbb{T}$-model is just a **group**.

(2) A model of $\mathbb{T}$ in $\widehat{\mathbb{C}}$ is a **presheaf of groups**.

(3) A model in $\text{Sh}(X)$ for a space X is a **sheaf of groups**.

Thm (Scott 1980) (Presheaf Completeness)

For any λ-theory $\mathbb{T}$ we have:

(1) $\mathbb{T} \vdash t : T \iff \mathcal{M} \models T$ f. all $\mathbb{C}$ and all $\mathbb{T}$ models $\mathcal{M}$ in $\hat{\mathbb{C}}$

(2) $\mathbb{T} \vdash s = t : E \iff \mathcal{M} \models s = t$ f. all $\mathbb{C}$ and all $\mathbb{T}$ models $\mathcal{M}$ in $\hat{\mathbb{C}}$.

Pf:

1. Build the syntactic CCC $\mathcal{C}_{\mathbb{T}}$, consisting of types & terms, mod equ's.
2. $\mathcal{C}_{\mathbb{T}}$ has a canonical model $\mathcal{U}$, consisting of the basic types & terms.
3. $\mathcal{U}$ is generic, in the sense:

$$\mathbb{T} \vdash t : T \iff \mathcal{U} \models T$$

$$\mathbb{T} \vdash s = t : E \iff \mathcal{U} \models s = t \,.$$

4. $\mathcal{C}_{\mathbb{T}}$ is the free CCC on a $\mathbb{T}$-model:

$\forall \mathcal{M} \, \exists ! \, \mathcal{M}^{\#}$:

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{\mathcal{M}^{\#}} & \mathcal{C} \quad \text{CCC} \\ \mathcal{U} & \longmapsto & \mathcal{M} \,. \end{array}$$

Next we need the following generalization of the main lemma from PL for $\downarrow : P \to \hat{P}$.

Lemma For any small cat $\mathbb{C}$, the cat

$$\hat{\mathbb{C}} = \mathrm{Set}^{\mathbb{C}^{op}}$$

of presheaves on $\mathbb{C}$ is cartesian closed, and the Yoneda embedding

$$y : \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$$

preserves any CCC structure in $\mathbb{C}$.

pf. For $P, Q \in \hat{\mathbb{C}}$ what should Q^P be?

$$\begin{aligned} (Q^P)c &\cong \hat{\mathbb{C}}(yc, Q^P) && \text{Yoneda} \\ &\cong \hat{\mathbb{C}}(yc \times P, Q) && \text{CCC} \end{aligned}$$

So let $Q^P := \hat{\mathbb{C}}(y(-) \times P, Q)$. ✓

Given $c, d \in \hat{\mathbb{C}}$,

$$\begin{aligned} y(d^c) &= \mathbb{C}(-, d^c) && \text{def} \\ &= \mathbb{C}(- \times c, d) && \text{CCC} \\ &\cong \hat{\mathbb{C}}(y(- \times c), yd) && \text{Yoneda} \\ &\cong \hat{\mathbb{C}}(y(-) \times yc, yd) && \text{UMP of } \times \\ &= (yd)^{yc}. \ ✓ && \text{def} \end{aligned}$$

To finish the proof of the thm:

if $\mathbb{T} \vdash t : T$, then $\mathcal{U} \models T$, namely

(*) $$[\![t]\!] : 1 \longrightarrow [\![T]\!] \quad \text{in } \mathcal{C}_{\mathbb{T}} \, .$$

Given any $\mathbb{T}$ model $\mathcal{M}$ in any $\hat{\mathbb{C}}$,
since $\mathcal{C}_{\mathbb{T}}$ is free on $\mathcal{U}$ we have:

$$\begin{aligned} \mathcal{C}_{\mathbb{T}} &\xrightarrow{\mathcal{M}^{\#}} \mathcal{C} \\ \mathcal{U} &\longmapsto \mathcal{M} \end{aligned} \, .$$

So from (*) we obtain:

$$\begin{tikzcd} \mathcal{M}^{\#} 1 \arrow[r, "\mathcal{M}^{\#} [\![t]\!]"] \arrow[d, phantom, "\cong"] & \mathcal{M}^{\#} [\![T]\!] \arrow[d, phantom, "\cong"] \\ 1 \arrow[r] & [\![T]\!]^{\mathcal{M}} \end{tikzcd}$$

Where the $\cong$'s are because $\mathcal{M}^{\#}$ is ccc.

Thus indeed:
$$\mathcal{M} \models T \, .$$

Conversely, if $\mathcal{M} \models T$ for all $\mathcal{M}$,
then in particular $\mathcal{U} \models T$.
Whence:
$$\mathbb{T} \vdash T \, ,$$

since $\mathcal{U}$ is generic.

(1) ✓

(2) %.

Finally, we can specialize from general cats $\mathbb{C}$ to posets K:

<u>Thm</u> (Kripke Completeness of λ-Calculus)

For any λ-theory $\mathbb{T}$ we have:

(1) $\mathbb{T} \vdash t : T \iff \mathcal{M} \models T$

f. all posets K and $\mathbb{T}$-models $\mathcal{M}$ in $\hat{K}$

(2) $\mathbb{T} \vdash s = t : E \iff \mathcal{M} \models s = t$

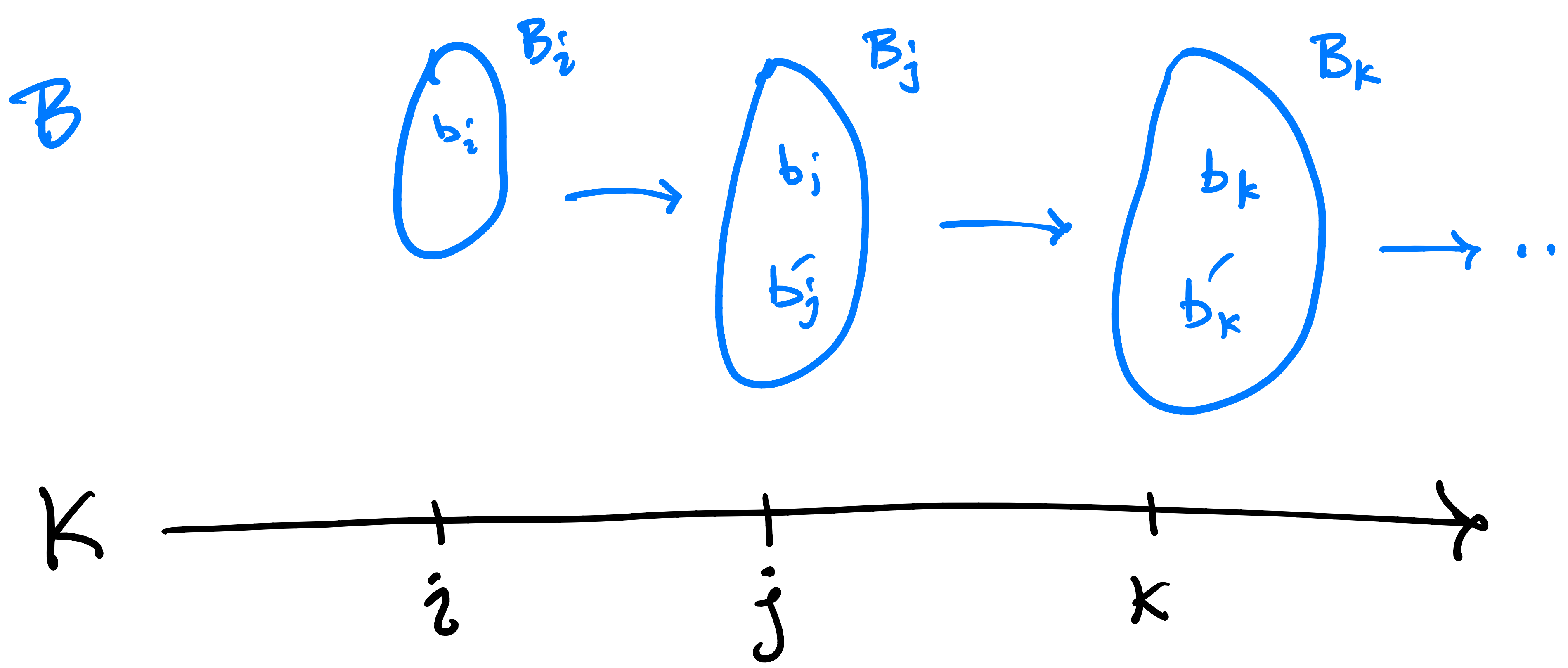
f. all posets K and $\mathbb{T}$-models $\mathcal{M}$ in $\hat{K}$.

The proof* uses a theorem from topos theory (due to Joyal & Tierney) to move from $\hat{\mathbb{C}}$ to $\widehat{\mathbb{C}^*}$ for a poset $\mathbb{C}^*$.

<u>Remark</u>: One can also go between the "Scott style" $j \in [\![A]\!]$ and the "Kripke style" $j \Vdash A$ for λ-theories (see AGH 2021).

* In (AR 2011).

What is a Kripke model of the λ-calculus?



$k \Vdash b : B$ f.a. $k \in K$

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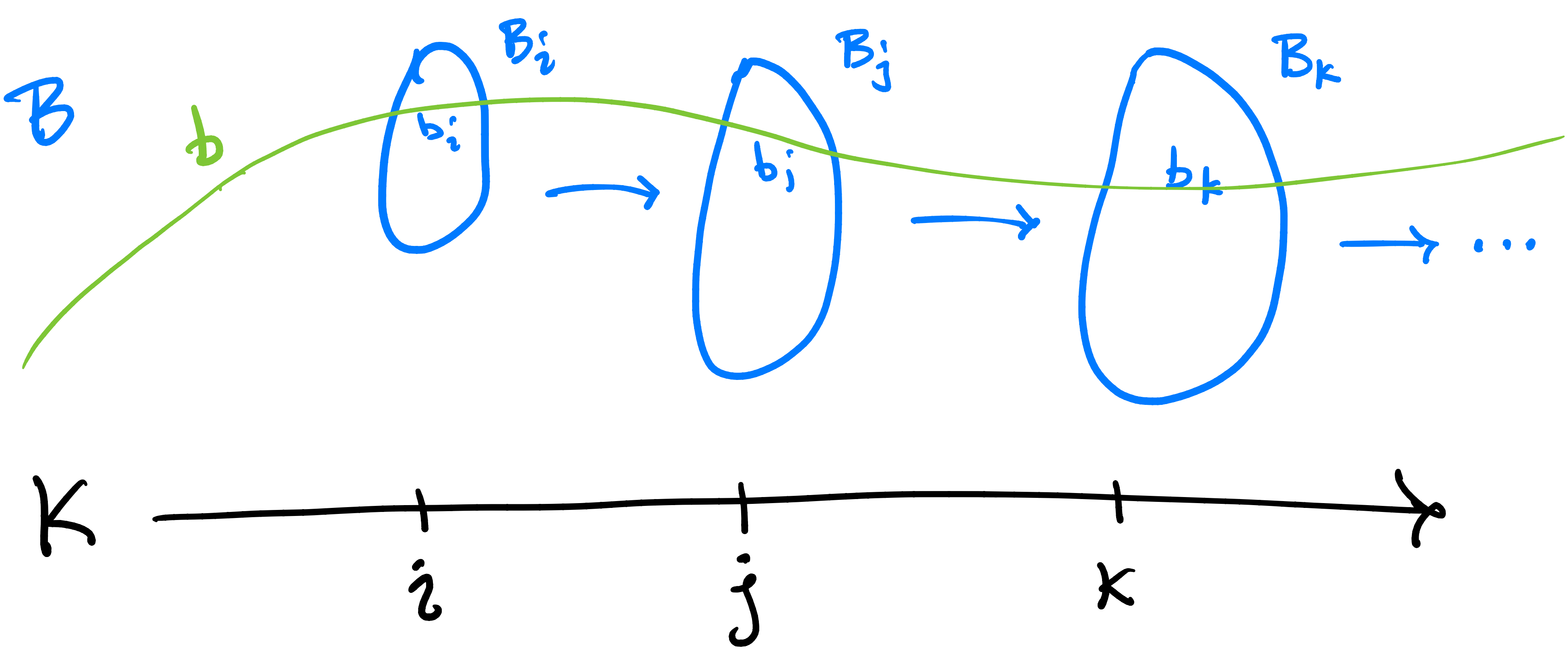

$k \Vdash b : B$ f.a. $k \in K$

$K \Vdash b : B$

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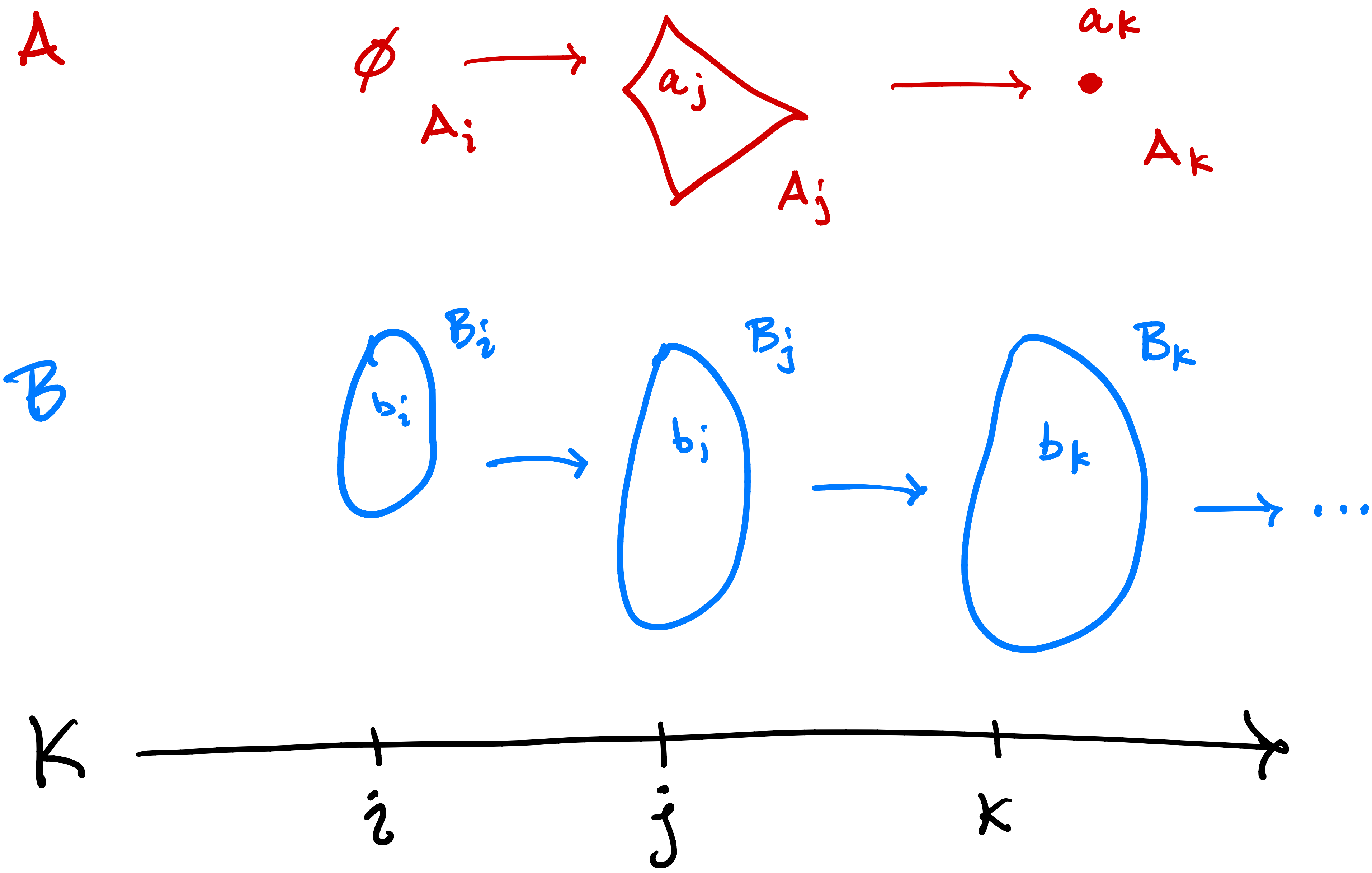
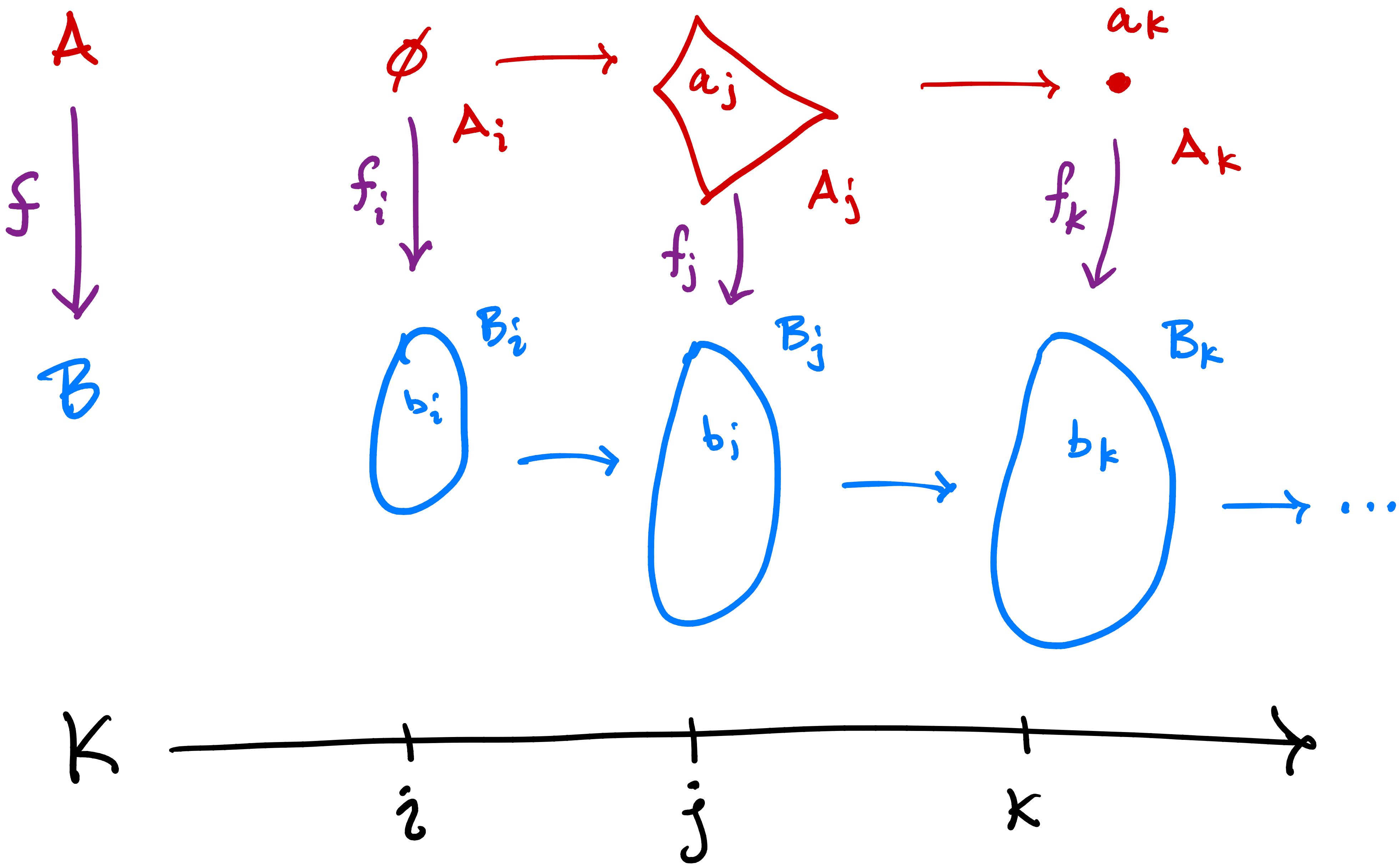


$$j \Vdash a : A \Rightarrow k \Vdash a : A$$

$$K \nVdash a : A$$

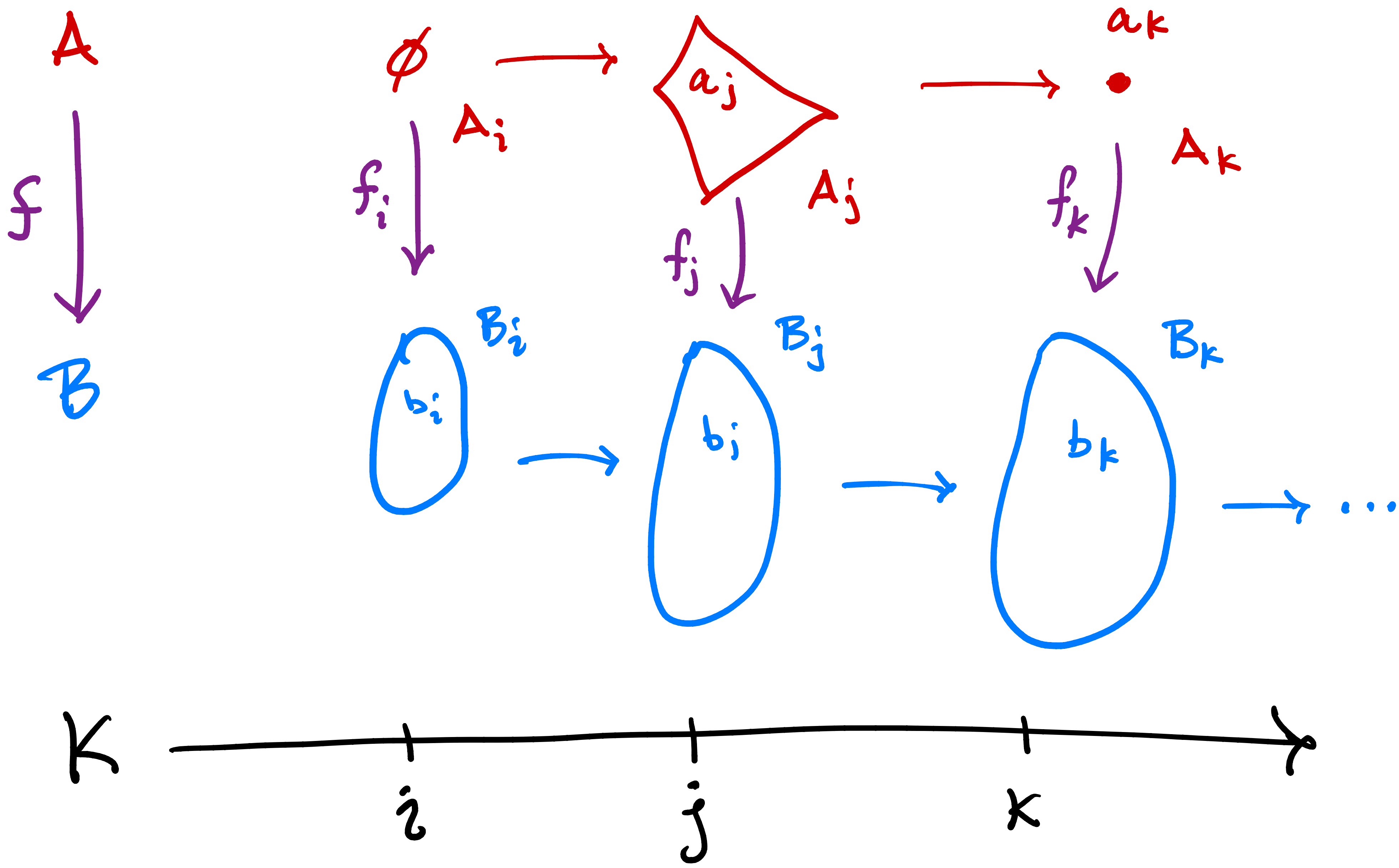
What is a Kripke model of the λ-calculus?



$$K \Vdash f : A \longrightarrow B$$

$$\Longleftrightarrow \quad j \Vdash f a = b \quad \text{f.a.}\ i \leq j$$

What is a Kripke model of the λ-calculus?



$$i \nVdash g : B \to A$$

$$\Rightarrow \quad K \nVdash g : B \to A$$

Open Problems

1) As in PL, one should be able to add $\mathbb{0}$ & $A+B$ to the λ-calculus and still get (both Scott $\hat{\mathbb{C}}$ and Kripke $\hat{K}$) completeness theorems.

2) The use of Joyal-Tierney to get from $\hat{\mathbb{C}}$ to $\hat{K}$ is probably overkill. It actually produces a sheaf model over a space $X_{\mathbb{C}}$, and then

$$K = \mathcal{O}X_{\mathbb{C}} \quad \text{(cf. A2000)}.$$

Perhaps there is a more direct proof, following the idea of the PL case?

3) Can one add $\mathbb{0}$ & $A+B$ in the topological case?

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