Kripke-Joyal Forcing for Martin-Löf Type Theory

Steve Awodey
jww/ Nicola Gambino and Sina Hazratpour

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Motivation

- Martin Löf type theory (MLTT) is common generalization of first-order logic (FOL) and the simply-typed lambda calculus, and is a powerful and expressive system of formal logic.
- It serves as the basis of Homotopy Type Theory, as well as several computer proof systems such as Agda, Coq, and Lean.
- It is a challenging problem to give semantics for MLTT that are both precise enough to strictly model the syntax and yet flexible enough to admit basic mathematical constructions.
- Kripke-Joyal forcing provides such semantics for both FOL and HOL and is here generalized to MLTT.

Kripke-Joyal forcing for FOL

Let $\mathbb{C}$ be a small category. For the topos of presheaves, write

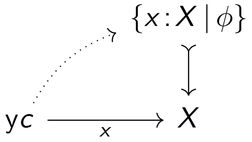
$$\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]\,.$$

We interpret a FOL formula $x{:}X \mid \phi$ over $X \in \widehat{\mathbb{C}}$ as a subobject,

$$\{x{:}X \mid \phi\} \rightarrowtail X\,.$$

Definition. Let $x : \mathsf{y}c \to X$. We say that x **forces** ϕ **at stage** c, if there is a factorization as on the right below.

$$c \Vdash \phi(x)$$



Kripke-Joyal forcing for FOL

Remark

- If $c \Vdash \phi(x)$ for *all* elements $x : \mathrm{y}c \to X$ we then have

$$\{x{:}X \mid \phi\} \cong X,$$

- If ϕ is *closed* we then have

$$\{\phi\} \cong 1\,.$$

- Then say that ϕ **holds** on $\mathbb{C}$ and write

$$\mathbb{C} \Vdash \phi\,.$$

Kripke-Joyal forcing for FOL

Key fact: We can recursively unwind the condition $c \Vdash \phi(x)$ according to the structure of ϕ,

$$
\begin{array}{lll}
c \Vdash \phi(x) \vee \psi(x) & \text{iff} & c \Vdash \phi(x) \text{ or } c \Vdash \psi(x) \\
c \Vdash \phi(x) \wedge \psi(x) & \text{iff} & c \Vdash \phi(x) \text{ and } c \Vdash \psi(x) \\
c \Vdash \phi(x) \Rightarrow \psi(x) & \text{iff} & d \Vdash \phi(xf) \text{ implies } d \Vdash \psi(xf), \text{ for all } f : d \to c \\
c \Vdash \exists y.\vartheta(x,y) & \text{iff} & c \Vdash \vartheta(x,y) \text{ for some } y : \mathrm{y}c \to Y \\
c \Vdash \forall y.\vartheta(x,y) & \text{iff} & d \Vdash \vartheta(xf,y) \text{ for all } f : d \to c \text{ and } y : \mathrm{y}d \to Y
\end{array}
$$

This provides a *quasi-mechanical* procedure for determining whether a formula holds in a model.

Kripke-Joyal forcing for MLTT

For MLTT we instead need to force a *dependent type*

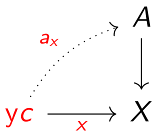
$$x : X \vdash A\,,$$

which is interpreted as a map $A \to X$ (an indexed family A_x), rather than a mere subobject $\{x : X \mid \phi\} \rightarrowtail X$.

This will require *forcing a term in context*,

$$c \Vdash a_x : A_x$$

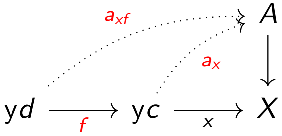
which is interpreted as a partial section.



Kripke-Joyal forcing for MLTT

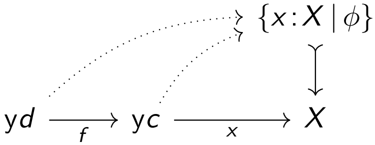
In order to force terms in stages $c \Vdash a_x : A_x$ we need a strict interpretation:

$$\frac{c \Vdash a_x : A_x}{d \Vdash a_{xf} : A_{xf}}$$



This is unlike the propositional case:

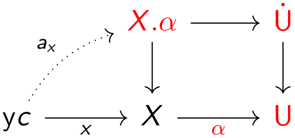
$$\frac{c \Vdash \phi(x)}{d \Vdash \phi(xf)}$$



Kripke-Joyal forcing for MLTT

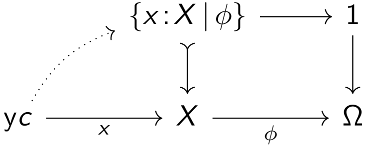
We will use a *universe* to ensure coherence.

$c \Vdash a_x : \alpha(x)$



This is like using the *subobject classifier* to interpret FOL.

$c \Vdash \phi(x)$



Kripke-Joyal forcing for MLTT

Proposition (Forcing terms)

For any type in context $X \vdash \alpha$ the following are equivalent.

- *there is a term t such that*

$$X \vdash t : \alpha$$

- *for all $x : \mathrm{y}c \to X$ there is given coherently t_x such that*

$$c \Vdash t_x : \alpha(x)\,.$$

Kripke-Joyal forcing for MLTT

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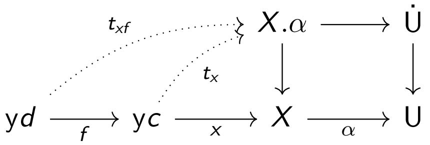
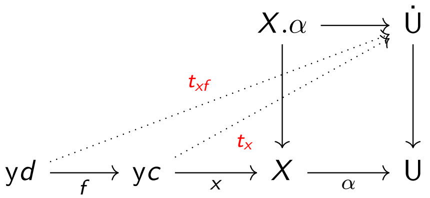
Kripke-Joyal forcing for MLTT

Proof. Coherence means that $t_{xf} = t_x \circ f$.



But these partial sections correspond to partial lifts of α,



Kripke-Joyal forcing for MLTT

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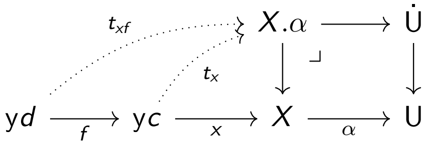

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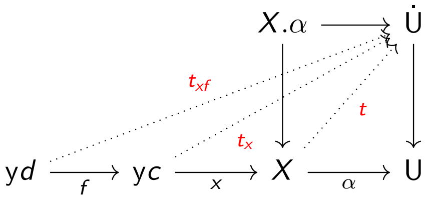

So the proof that $X \vdash t : \alpha$ is complete by Yoneda.

Outline

1. The universe $\dot{\mathsf{U}} \to \mathsf{U}$

For κ sufficiently large, define small categories

$$
\begin{array}{lll}
\mathsf{Set}_\kappa \hookrightarrow \mathsf{Set} & \qquad & \textit{small sets} \\
\dot{\mathsf{Set}}_\kappa \hookrightarrow \dot{\mathsf{Set}} & \qquad & \textit{small pointed sets}
\end{array}
$$

and presheaves

$$
\begin{aligned}
\dot{\mathsf{U}} &= \mathsf{Cat}\big(\mathbb{C}/\_^{\mathsf{op}}, \dot{\mathsf{Set}}_\kappa\big) \\
\mathsf{U} &= \mathsf{Cat}\big(\mathbb{C}/\_^{\mathsf{op}}, \mathsf{Set}_\kappa\big)
\end{aligned}
$$

1. The universe $\dot{\mathsf{U}} \to \mathsf{U}$

The action on $(P : \mathbb{C}/_c{}^{\mathrm{op}} \to \mathsf{Set}_\kappa) \in \mathsf{U}c$ is by precomposition.

$$
\begin{array}{ccccc}
c & \qquad & \mathbb{C}/_c{}^{\mathrm{op}} & \xrightarrow{P} & \mathsf{Set}_\kappa \\
\uparrow & & \uparrow & \nearrow & \\
d & & \mathbb{C}/_d{}^{\mathrm{op}} & &
\end{array}
$$

Naturality of $\dot{\mathsf{U}} \to \mathsf{U}$ is then automatic.

$$
\begin{array}{ccccccccc}
c & \qquad & \mathbb{C}/_c{}^{\mathrm{op}} & \xrightarrow{P} & \dot{\mathsf{Set}}_\kappa & \qquad & \dot{\mathsf{U}}c & \longrightarrow & \mathsf{U}c \\
\uparrow & & \uparrow & & \downarrow & & \downarrow & & \downarrow \\
d & & \mathbb{C}/_d{}^{\mathrm{op}} & \longrightarrow & \mathsf{Set}_\kappa & & \dot{\mathsf{U}}d & \longrightarrow & \mathsf{U}d
\end{array}
$$

1. The universe $\dot{\mathsf{U}} \to \mathsf{U}$

Definition (Small presheaves)

A presheaf A is *small* if all its values are small.
A map $A \to X$ is *small* if all its fibers A_x are small.

$$\begin{array}{ccc} A_x & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathsf{y}c & \xrightarrow[x]{} & X \end{array}$$

Lemma ($\dot{\mathsf{U}} \to \mathsf{U}$ classifies small maps)

For small $A \to X$ there is an $\alpha : X \to \mathsf{U}$ and a pullback

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathsf{U}} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\alpha]{} & \mathsf{U} \end{array}$$

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Lemma ($\dot{\mathsf{U}} \to \mathsf{U}$ classifies small maps)

For small $A \to X$ there is a canonical $\alpha : X \to \mathsf{U}$ and a chosen pullback

$$\begin{array}{ccccc} A & \xrightarrow{\cong} & X.\alpha & \longrightarrow & \dot{\mathsf{U}} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[=]{} & X & \xrightarrow[\alpha]{} & \mathsf{U} \end{array}$$

2. The natural model of MLTT

Let $f : Y \to X$ and consider the two-pullbacks diagram arising from substitution.

$$\frac{X \vdash \alpha}{Y \vdash \alpha f} \qquad \begin{array}{ccccc} Y.\alpha f & \longrightarrow & X.\alpha & \longrightarrow & \dot{\mathsf{U}} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow[f]{} & X & \xrightarrow[\alpha]{} & \mathsf{U} \end{array}$$

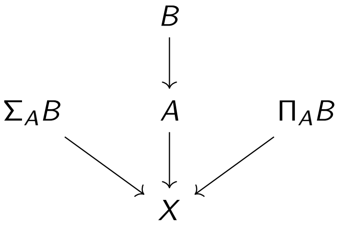
The pullback functor f^* is thus modeled by precomposition of classifying maps into U.

$$\begin{array}{c} Y \\ \downarrow f \\ X \end{array} \qquad \begin{array}{ccccc} \mathrm{Hom}(Y, \mathsf{U}) & \xrightarrow{\sim} & \mathcal{S}/_Y & \hookrightarrow & \mathcal{E}/_Y \\ {\scriptstyle -\circ f}\uparrow & & {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{Hom}(X, \mathsf{U}) & \xrightarrow[\sim]{} & \mathcal{S}/_X & \hookrightarrow & \mathcal{E}/_X \end{array}$$

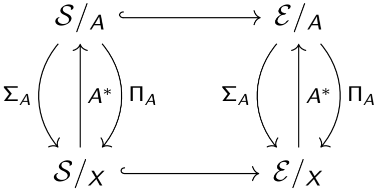
2. The natural model of MLTT

For small $A \to X$ the adjoint functors

$$\Sigma_A B \dashv A^* \dashv \Pi_A B$$

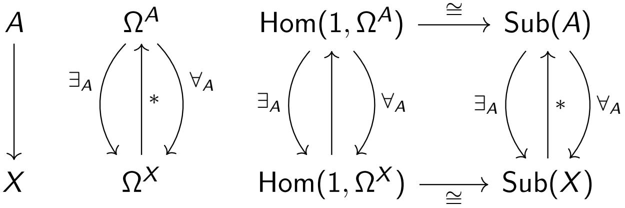


all preserve the small maps,



2. The natural model of MLTT

These type formers Σ, Π are induced by structure on $\dot{\mathsf{U}} \to \mathsf{U}$, in the same way that the quantifiers on subobjects are induced by maps on powerobjects.



In more detail ...

2. The natural model of MLTT

The polynomial object

$$P\mathsf{U} = \sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

classifies *types in context*:

$$\frac{(A, B) : \Gamma \longrightarrow P\mathsf{U}}{\Gamma.A \vdash B}$$

Similarly, the object

$$P\dot{\mathsf{U}} = \sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]}$$

classifies *terms in context* $\Gamma.A \vdash b : B$.

2. The natural model of MLTT

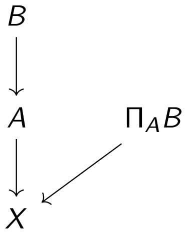
Proposition

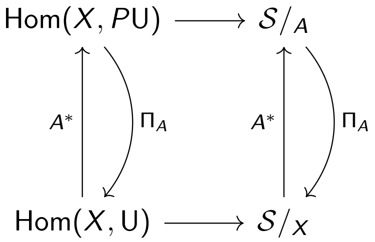
The universe $\dot{\mathsf{U}} \to \mathsf{U}$ *models the rules for products just if there are maps* λ, Π *making a pullback diagram.*

$$\begin{array}{ccc} P\dot{\mathsf{U}} & \xrightarrow{\quad\lambda\quad} & \dot{\mathsf{U}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ P\mathsf{U} & \xrightarrow[\quad\Pi\quad]{} & \mathsf{U} \end{array}$$

2. The natural model of MLTT

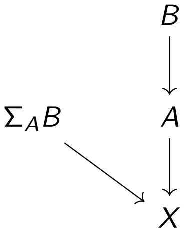
The right adjoint $A^* \dashv \Pi_A B$ is induced by composing classifying maps with $\Pi : P\mathsf{U} \to \mathsf{U}$.

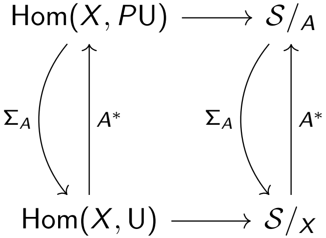




2. The natural model of MLTT

There is a similar structure $\Sigma : P\mathrm{U} \to \mathrm{U}$ inducing the left adjoint $\Sigma_A \dashv A^*$.





2. The natural model of MLTT

Proposition

The natural model structure on the universe provides a strict interpretation of MLTT.

$$
\begin{array}{ccc}
Q & \xrightarrow{\text{pair}} & \dot{\mathsf{U}} \\
\downarrow & & \downarrow \\
P\mathsf{U} & \xrightarrow[\Sigma]{} & \mathsf{U}
\end{array}
\qquad\qquad
\begin{array}{ccc}
P\dot{\mathsf{U}} & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\
\downarrow & & \downarrow \\
P\mathsf{U} & \xrightarrow[\Pi]{} & \mathsf{U}
\end{array}
$$

We use this structure to give forcing conditions for Σ and Π at $x : \mathsf{y}c \to X$, as in

$$c \Vdash t : \Sigma_{y:\alpha(x)}\beta(x, y)$$

$$c \Vdash t : \Pi_{y:\alpha(x)}\beta(x, y)$$

3. The Kripke-Joyal forcing rules

Theorem

Let $X \in \widehat{\mathbb{C}}$ and $\alpha : X \to \mathsf{U}$ and $\beta : X.\alpha \to \mathsf{U}$.
For all $x : \mathsf{y}c \to X$, we have

$$
\begin{array}{lll}
c \Vdash t : 0 & \textit{iff} & t \neq t \\
c \Vdash t : 1 & \textit{iff} & t = * \\
c \Vdash t : (\alpha + \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{or} \quad c \Vdash b : \beta(x) \\
c \Vdash t : (\alpha \times \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{and} \quad c \Vdash b : \beta(x) \\
c \Vdash t : (\Sigma_\alpha \beta)(x) & \textit{iff} & c \Vdash a : \alpha(x) \quad \textit{and} \quad c \Vdash b : \beta(x, a) \\
c \Vdash t : (\Pi_\alpha \beta)(x) & \textit{iff} & \textit{for all } f : d \to c \textit{ and } d \Vdash a : \alpha(xf) \\
 & & \textit{there's } d \Vdash b_{f,a} : \beta(xf, a) \textit{ coherently}
\end{array}
$$

3. Kripke-Joyal forcing rules

Definition

Let $X \in \widehat{\mathbb{C}}$ and $\alpha : X \to \mathrm{U}$ a type over X.
We say that $\mathbb{C}$ *forces a term of type* α,

$$\mathbb{C} \Vdash X \vdash t : \alpha$$

if for all $c \in \mathbb{C}$ and all $x : \mathrm{y}c \to X$, there is given coherently

$$c \Vdash t : \alpha(x)$$

4. The completeness theorem

Theorem (A-Gambino-Hazratpour)

Let C *be a closed type in* MLTT *with the type forming operations*

$$0,\ 1,\ X,\ A+B,\ A\times B,\ A\to B,\ \Sigma_A B,\ \Pi_A B,\ s=_A t\,.$$

There is a closed term $\vdash t : C$ *if, and only if, for all categories* $\mathbb{C}$ *and all presheaves* X *on* $\mathbb{C}$*, one has* $\mathbb{C} \Vdash t : C$*. Briefly,*

$$\mathsf{MLTT} \vdash t : C \quad \textit{iff} \quad \mathbb{C} \Vdash t : C \quad \textit{for all } \mathbb{C} \textit{ and } X.$$

Moreover, it suffices to assume that $\mathbb{C}$ *is a poset.*

4. The completeness theorem

Proof. Let $P = \mathcal{O}X_{\mathbb{T}}$, where $\mathbb{T}$ is the classifying category of MLTT, and $p : \mathsf{Sh}(X_{\mathbb{T}}) \twoheadrightarrow \widehat{\mathbb{T}}$ is the spatial cover.

There are LCCC embeddings:

$$\mathbb{T} \xhookrightarrow{\mathsf{y}} \widehat{\mathbb{T}} \xhookrightarrow{p^*} \mathsf{Sh}(X_{\mathbb{T}}) \hookrightarrow \widehat{\mathcal{O}X_{\mathbb{T}}}.$$

So we have:

$$\begin{array}{rcll} \mathsf{MLTT} \vdash t : C & \iff & 1 \xrightarrow{t} C & \mathbb{T} \\ & \iff & 1 \cong \mathsf{y}1 \xrightarrow{\mathsf{y}t} \mathsf{y}C \cong [\![C]\!]^{\mathbb{T}} & \widehat{\mathbb{T}} \\ & \iff & 1 \cong p^*\mathsf{y}1 \xrightarrow{p^*\mathsf{y}t} p^*\mathsf{y}C \cong [\![C]\!]^{X_{\mathbb{T}}} & \mathsf{Sh}(X_{\mathbb{T}}) \\ & \iff & \mathcal{O}X_{\mathbb{T}} \Vdash t : C & \square \end{array}$$

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