# Algebraic Type Theory

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# Outline

1. Natural models of type theory
2. Type formers and polynomials
3. Strictifying homotopical models
4. A polynomial monad
5. Martin-Löf algebras

# 1. Martin-Löf type theory

The system of dependent type theory to be modeled consists of:

**Types**: $A, B, C, \ldots$

**Terms**: $x : A,\ b : B,\ c : C, \ldots$

**Contexts of variables**: $(x : A, y : B, \ldots), \ldots, \Gamma, \Delta, \ldots$

**Dependent types and terms**: $x : A \vdash b : B, \ldots$

**Substitutions**: $\sigma : \Delta \to \Gamma, \ldots$

**Type formers**: $\sum_{x:A} B,\ \prod_{x:A} B,\ \mathsf{Id}_A(a, b),\ \ldots$

# 1. Martin-Löf type theory

**Contexts**:

$$\frac{}{[\cdot] \vdash} \qquad \frac{\Gamma \vdash C}{\Gamma, z : C \vdash}$$

**Sums**:

$$\frac{x : A \vdash B}{\sum_{x:A} B} \qquad \frac{a : A \qquad b : B(a)}{\langle a, b \rangle : \sum_{x:A} B}$$

$$\frac{c : \sum_{x:A} B}{\mathsf{fst}\, c : A} \qquad \frac{c : \sum_{x:A} B}{\mathsf{snd}\, c : B(\mathsf{fst}\, c)}$$

$$\mathsf{fst}\langle a, b \rangle = a : A \qquad \mathsf{snd}\langle a, b \rangle = b : B$$

$$\langle \mathsf{fst}\, c, \mathsf{snd}\, c \rangle = c : \sum_{x:A} B(x)$$

# 1. Martin-Löf type theory

**Products**:

$$\frac{x : A \vdash B}{\prod_{x:A} B} \qquad \frac{x : A \vdash b : B}{\lambda x.b : \prod_{x:A} B}$$

$$\frac{a : A \qquad f : \prod_{x:A} B}{fa : B(a)}$$

$$x : A \vdash (\lambda x.b)x = b : B$$

$$\lambda x.fx = f : \prod_{x:A} B$$

**Substitution**:

$$\frac{\sigma : \Delta \to \Gamma \qquad \Gamma \vdash a : A}{\Delta \vdash a\sigma : A\sigma}$$

# 1. Natural models of type theory

## Definition

A natural transformation $p : \dot{\mathsf{U}} \to \mathsf{U}$ of presheaves on a category $\mathbb{C}$ is **presentable** if the pullback along any element $x : \mathsf{y}C \to X$ is representable.

$$\begin{array}{ccc} \mathsf{y}D & \xrightarrow{\;y\;} & \dot{\mathsf{U}} \\ \big\downarrow & \lrcorner & \big\downarrow p \\ \mathsf{y}C & \xrightarrow[\;x\;]{} & \mathsf{U} \end{array}$$

If $\mathbb{C}$ has finite limits, $p : \dot{\mathsf{U}} \to \mathsf{U}$ is presentable iff it is **tiny** in the sense that the pushforward functor

$$p^* \dashv p_* : \widehat{\mathbb{C}}/\dot{\mathsf{U}} \longrightarrow \widehat{\mathbb{C}}/\mathsf{U}$$

has a *right* adjoint:

$$p_! \dashv p^* \dashv p_* \dashv p^!$$

# 1. Natural models of type theory

### Proposition (A., Fiore 2013)

*A presentable natural transformation is the same thing as a* **category with families** *in the sense of Dybjer.*

# 1. Natural models as CwFs

The objects and arrows $\sigma : \Delta \to \Gamma$ of $\mathbb{C}$ are the **contexts and substitutions**.

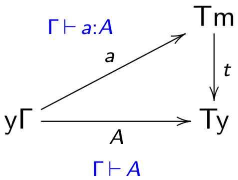
The presheaves are the **types** and **terms in context**,

$$\mathsf{Ty}, \mathsf{Tm} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}\,,$$

along with a "typing" map $t : \mathsf{Tm} \to \mathsf{Ty}$.

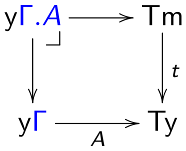
# 1. Natural models as CwFs

We then interpret:



# 1. Natural models as CwFs

For the **context extension** $\Gamma.A \to A$ we use the fact that $t$ is presentable.



# 2. The type formers and polynomials

Recall that any map $p : \dot{\mathsf{U}} \to \mathsf{U}$ in an LCCC such as $\widehat{\mathbb{C}}$ determines a **polynomial endofunctor**

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\quad P \quad} & \widehat{\mathbb{C}} \\ {\scriptstyle \Delta_{\dot{\mathsf{U}}}}\big\downarrow & & \big\uparrow{\scriptstyle \Sigma_{\mathsf{U}}} \\ \widehat{\mathbb{C}}/\dot{\mathsf{U}} & \xrightarrow[\quad \Pi_p \quad]{} & \widehat{\mathbb{C}}/U \end{array}$$

by

$$\begin{array}{ccccc} X & \longleftarrow & X \times \dot{\mathsf{U}} & & PX \\ & & \big\downarrow & & \big\downarrow \\ & & \dot{\mathsf{U}} & \xrightarrow[\quad p \quad]{} & \mathsf{U} \end{array}$$

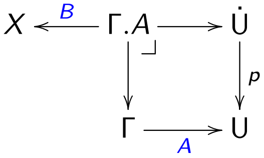
which may be written

$$PX \;=\; \sum_{A\,:\,\mathsf{U}} X^A \,.$$

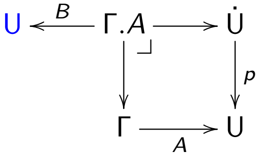
# 2. The type formers and polynomials

### Lemma

*Maps* $\Gamma \to PX$ *correspond naturally to pairs* $(A, B)$ *where*




The object $P\mathsf{U}$ therefore classifies **types in context** $\Gamma.A \vdash B$



# 2. The type formers: $\Pi$

### Proposition

*The model $p : \dot{\mathsf{U}} \to \mathsf{U}$ has $\Pi$-types just if there are maps $\lambda$ and $\Pi$ making the following a pullback.*

$$\begin{array}{ccc} P\dot{\mathsf{U}} & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ {\scriptstyle Pp}\downarrow & & \downarrow{\scriptstyle p} \\ P\mathsf{U} & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

# 2. The type formers: $\Pi$

## Proposition

*The model $p : \dot{\mathsf{U}} \to \mathsf{U}$ has $\Pi$-types just if there are maps $\lambda$ and $\Pi$ making the following a pullback.*

*Proof:*

$A \vdash b : B$ $\qquad\qquad$ $\lambda_A b$

$$
\begin{array}{ccc}
P\dot{\mathsf{U}} & \xrightarrow{\ \lambda\ } & \dot{\mathsf{U}} \\
\big\downarrow & & \big\downarrow{\scriptstyle p} \\
P\mathsf{U} & \xrightarrow[\ \Pi\ ]{} & \mathsf{U}
\end{array}
$$

$A \vdash B$ $\qquad\qquad$ $\Pi_A B$

# 2. The type formers: $\Sigma$

### Proposition

*The model $p : \dot{\mathsf{U}} \to \mathsf{U}$ has $\Sigma$-types just if there are maps* $(\mathsf{pair}, \Sigma)$ *making the following a pullback*

$$\begin{array}{ccc} Q & \xrightarrow{\ \mathsf{pair}\ } & \dot{\mathsf{U}} \\ {\scriptstyle p.p}\downarrow & & \downarrow{\scriptstyle p} \\ P\mathsf{U} & \xrightarrow[\ \Sigma\ ]{} & \mathsf{U} \end{array}$$

*where $p.p : Q \to P\mathsf{U}$ is such that $P_{p.p} = P_p \circ P_p$.*

# 2. The type formers: Identity

To model identity types, take $(\mathsf{i}, \mathsf{Id})$ making the following commute.

$$\begin{CD} \dot{\mathsf{U}} @>{\mathsf{i}}>> \dot{\mathsf{U}} \\ @VVV @VVV \\ \dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}} @>>{\mathsf{Id}}> \mathsf{U} \end{CD}$$

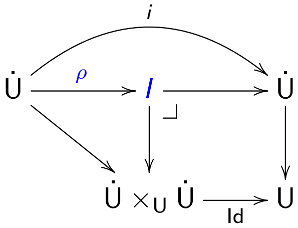
This models the formation and introduction rules.

$$x, y : A \vdash \mathsf{Id}_A(x, y)$$

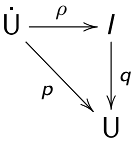
$$x : A \vdash \mathsf{i}x : \mathsf{Id}_A(x, x)$$

# 2. The type formers: Identity

Next, take a pullback to get an object I and a map $\rho : \dot{\mathsf{U}} \to \mathsf{I}$,




which commutes with the indicated projections to U.



# 2. The type formers: Identity

The map $\rho : \dot{\mathrm{U}} \to \mathrm{I}$ gives a natural transformation,

$$\rho^* : P_q \to P_p$$

evaluating which at $p : \dot{\mathrm{U}} \to \mathrm{U}$ gives a commutative square,

$$\begin{array}{ccc} P_q\dot{\mathrm{U}} & \xrightarrow{\rho^*_{\dot{\mathrm{U}}}} & P_p\dot{\mathrm{U}} \\ {\scriptstyle P_q p}\big\downarrow & & \big\downarrow{\scriptstyle P_p p} \\ P_q\dot{\mathrm{U}} & \xrightarrow[\rho^*_{\mathrm{U}}]{} & P_p\mathrm{U} \end{array}$$

A **weak pullback structure** is a section of the comparison map.

$$P_q\dot{\mathrm{U}} \longrightarrow P_q\dot{\mathrm{U}} \times_{P_p\mathrm{U}} P_p\dot{\mathrm{U}} \qquad \left(J : P_q\dot{\mathrm{U}} \times_{P_p\mathrm{U}} P_p\dot{\mathrm{U}} \dashrightarrow P_q\dot{\mathrm{U}}\right)$$

# 2. The type formers: Identity

## Proposition (Garner)

*The model* $p : \dot{\mathsf{U}} \to \mathsf{U}$ *has* **intensional** *identity types just if there are maps* $(\mathsf{i}, \mathsf{Id})$ *making the following commute*

$$\begin{array}{ccc} \dot{\mathsf{U}} & \xrightarrow{\quad \mathsf{i} \quad} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow \\ \dot{\mathsf{U}} \times_{\mathsf{U}} \dot{\mathsf{U}} & \xrightarrow[\quad \mathsf{Id} \quad]{} & \mathsf{U} \end{array}$$

*together with a weak pullback structure* $\mathsf{J}$ *for the resulting comparison naturality square.*

This models the standard elimination and computation rules.

$$\frac{x : A \vdash c : C(\rho x)}{x, y : A, z : \mathsf{Id}_A(x, y) \vdash \mathsf{J}_c : C} \qquad x : A \vdash \mathsf{J}_c(\rho x) = c : C(\rho x)$$

# 3. Strictifying homotopical models

**Theorem (A.-Garner 2016, cf. Lumsdaine-Warren 2015)**
*Let $(\mathbb{C}, \mathcal{F})$ be a $\Pi$-tribe in the sense of Joyal. Then the coproduct of the $\mathcal{F}$-maps in $\widehat{\mathbb{C}}$,*

$$\coprod_{f \in \mathcal{F}} \mathsf{y}\mathrm{dom} f \xrightarrow{\coprod_{f \in \mathcal{F}} \mathsf{y} f} \coprod_{f \in \mathcal{F}} \mathsf{y}\mathrm{cod} f$$

*is a natural model with $\Sigma$, $\Pi$ and $\mathsf{Id}$ types.*

For example, any homotopical model in a right-proper Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category of presheaves $\widehat{\mathbb{C}}$ has a **strictification**:

$$p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$$

# 4. A polynomial monad

Consider the rules for a **unit type** T.

$$\frac{}{\vdash \mathsf{T}} \qquad\qquad \frac{}{\vdash * : \mathsf{T}} \qquad\qquad \frac{}{x : \mathsf{T} \vdash x = * : \mathsf{T}}$$

## Proposition

*A model* $p : \dot{\mathsf{U}} \to \mathsf{U}$ *has a unit type just if there are maps* $(*, \mathsf{T})$ *making the following a pullback.*

$$\begin{array}{ccc} 1 & \xrightarrow{\;*\;} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[\;\mathsf{T}\;]{} & \mathsf{U} \end{array}$$

# 4. A polynomial monad

The pullback squares for T and Σ

$$
\begin{array}{ccc}
1 & \xrightarrow{\ *\ } & \dot{\mathsf{U}} \\
\downarrow & & \downarrow{\scriptstyle p} \\
1 & \xrightarrow[\ \mathsf{T}\ ]{} & \mathsf{U}
\end{array}
\qquad\qquad
\begin{array}{ccc}
Q & \xrightarrow{\ \mathsf{pair}\ } & \dot{\mathsf{U}} \\
{\scriptstyle p.p}\downarrow & & \downarrow{\scriptstyle p} \\
P\mathsf{U} & \xrightarrow[\ \Sigma\ ]{} & \mathsf{U}
\end{array}
$$

determine cartesian natural transformations between the corresponding polynomial endofunctors.

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

# 4. A polynomial monad

Summarizing:

Theorem (A.-Newstead 2018)

*A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ has* $\top$ *and* $\Sigma$ *types just if the associated polynomial endofunctor $P$ has the structure of a cartesian pseudomonad.*

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

# 4. A polynomial monad

The monad laws express the following type isomorphisms.

| | |
|---|---|
| $\sigma \circ P\sigma = \sigma \circ \sigma_P$ | $\sum_{a:A} \sum_{b:B(a)} C(a,b) \;\cong\; \sum_{(a,b):\sum_{a:A} B(a)} C(a,b)$ |
| $\sigma \circ P\tau = 1$ | $\sum_{a:A} 1 \;\cong\; A$ |
| $\sigma \circ \tau_P = 1$ | $\sum_{x:1} A \;\cong\; A$ |

# 4. A polynomial monad

The pullback square for $\Pi$

$$\begin{array}{ccc} P\dot{\mathsf{U}} & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ {\scriptstyle Pp}\downarrow & & \downarrow{\scriptstyle p} \\ P\mathsf{U} & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

is an algebra structure

$$\pi : P^{\downarrow}p \Rightarrow p$$

for the lifted endofunctor $P^{\downarrow} : \widehat{\mathbb{C}}^{\downarrow} \to \widehat{\mathbb{C}}^{\downarrow}$ on the cartesian arrow category $\widehat{\mathbb{C}}^{\downarrow}$.

# 4. A polynomial monad

The monad algebra laws also correspond to type isomorphisms.

| | |
|---|---|
| $\pi \circ P\pi \;=\; \pi \circ \sigma$ | $\prod_{a:A} \prod_{b:B(a)} C(a,b) \;\cong\; \prod_{(a,b):\sum_{a:A} B(a)} C(a,b)$ |
| $\pi \circ \tau \;=\; 1$ | $\prod_{x:1} A \;\cong\; A$ |

# 5. Martin-Löf algebras

We can use the foregoing to axiomatize models of MLTT.

### Definition

A **Martin-Löf algebra** in an lccc $\mathcal{E}$ is a tiny map $p : \dot{\mathsf{U}} \to \mathsf{U}$ equipped with pullback squares:

$$
\begin{CD}
1 @>>> \dot{\mathsf{U}} \\
@VVV \top @VV{p}V \\
1 @>>> \mathsf{U}
\end{CD}
\qquad\qquad
\begin{CD}
Q @>>> \dot{\mathsf{U}} \\
@V{p.p}VV \Sigma @VV{p}V \\
P\mathsf{U} @>>> \mathsf{U}
\end{CD}
\qquad\qquad
\begin{CD}
P\dot{\mathsf{U}} @>>> \dot{\mathsf{U}} \\
@V{Pp}VV \Pi @VV{p}V \\
P\mathsf{U} @>>> \mathsf{U}
\end{CD}
$$

# 5. Martin-Löf algebras

By the strictification theorem, a homotopical model of MLTT in a right proper model category $\mathcal{E}$ determines an ML-algebra $p : \dot{\mathsf{U}} \to \mathsf{U}$, *which also has identity types*.

Corollary

*If* $p : \dot{\mathsf{U}} \to \mathsf{U}$ *is univalent, then the monad and algebra structures on the associated polynomial* $P : \mathcal{E} \to \mathcal{E}$ *satisfy the monad and algebra laws up to identity.*

Next: morphisms of M-L algebras, free M-L algebras, etc.

# References

1. Awodey, S. (2017) Natural models of homotopy type theory, MSCS 28(2). arXiv:1406.3219
2. Awodey, S. and C. Newstead (2018) Polynomial pseudomonads and dependent type theory. arXiv:1802.00997
3. P.L. Lumsdaine and M. Warren (2015) The local universes model, ACM Trans. Comp. Log. arXiv:1411.1736
4. Newstead, C. (2018) Algebraic Models of Dependent Type Theory, CMU PhD thesis. arXiv:2103.06155