Cartesian cubical model categories

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Background

- There has recently been work on **cubical** homotopy theory.
- It is related to **homotopy type theory** which is being used for computerized proof checking.
- The cubes used for this are **closed under finite products**.
- This model of homotopy was also proposed by Lawvere who stressed the **tinyness of the geometric interval** $\mathbb{I}$.
- The tinyness of $\mathbb{I}$ is also used in the current theory.
The **Cartesian cube category** \( \Box \) is the opposite of the category \( \mathbb{B} \) of finite, strictly bipointed sets,

\[
\Box : = \mathbb{B}^{\text{op}}.
\]

Thus \( \Box \) is the **Lawvere theory of bipointed objects**: the free finite product category with a bipointed object \([0] \twoheadrightarrow [1]\).

The **Cartesian cubical sets** is the category of presheaves on \( \Box \),

\[
cSet = \text{Set}^{\Box^{\text{op}}}
\]

Thus cSet consists of all **covariant** functors \( \mathbb{B} \rightarrow \text{Set} \).
The tiny interval $\mathbb{I}$

The 1-cube $[1]$ represents the cubical set that forgets the points,

$$\mathbb{I} := \mathbb{B}([1], -) : \mathbb{B} \longrightarrow \text{Set}.$$ 

It generates $\text{cSet}$ under finite products and colimits.

The two points $1 \Rightarrow \mathbb{I}$ have a trivial intersection.

$$\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{I}
\end{array}$$

This is the universal interval in a topos.

It provides a good cylinder $X + X \hookrightarrow \mathbb{I} \times X$ for every object $X$, and a good path object $X^{\mathbb{I}} \rightarrow X \times X$ for every fibrant object $X$. 
The main result

Theorem (A. 2023)

There is a Quillen model structure \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) on \(cSet\) where:

- **the cofibrations** \(\mathcal{C}\) are an axiomatized class of monos,
- **the fibrations** \(\mathcal{F}\) are those \(f : X \to Y\) for which
  \[
  (f^{\mathbb{I} \times \mathbb{I}}, \text{eval}) : X^{\mathbb{I} \times \mathbb{I}} \to (Y^{\mathbb{I} \times \mathbb{I}}) \times_Y X
  \]
  lifts on the right against all cofibrations,
- **the weak equivalences** \(\mathcal{W}\) are those \(f : X \to Y\) for which
  \(K^f : K^Y \to K^X\) is bijective under \(\pi_0\) whenever \(K\) is fibrant.
The construction of \((\mathcal{C}, \mathcal{W}, \mathcal{F})\)

The **proof** of the theorem

- uses ideas from **type theory**,  
- including the **univalence axiom** of Voevodsky,  
- is **axiomatized** in terms of:
  1. a classifier \(\Phi \to \Omega\) for the cofibrations,  
  2. a tiny interval \(1 \Rightarrow \mathbb{I}\),  
  3. a universal small map \(\dot{V} \to V\),  
- applies in several different cases.
The model structure \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) is constructed in 3 steps:

1. \(\Phi\) is used to determine a wfs \((\mathcal{C}, \text{TFib})\),
2. \(\mathbb{I}\) is used to determine a wfs \((\text{TCof}, \mathcal{F})\) with \(\text{TFib} \subseteq \mathcal{F}\),
3. \(V\) is used to show 3-for-2 for \(\mathcal{W} := \text{TFib} \circ \text{TCof}\).
1. The cofibration wfs \((\mathcal{C}, \text{TFib})\)

The **cofibrations** \(\mathcal{C}\) are the monos \(C' \hookrightarrow C\) classified by \(t : 1 \hookrightarrow \Phi\).

\[
\begin{array}{ccc}
C' & \rightarrow & 1 \\
\downarrow & & \downarrow t \\
C & \rightarrow & \Phi \\
\end{array}
\]

The **trivial fibrations** TFib are the maps \(T \rightarrow X\) that lift against the cofibrations.

\[
\begin{array}{ccc}
C' & \rightarrow & T \\
\downarrow & & \downarrow \\
C & \rightarrow & X \\
\end{array}
\]

\(\mathcal{C}' =: \text{TFib}\)
1. The cofibration wfs $(\mathcal{C}, \text{TFib})$

Proposition
$(\mathcal{C}, \text{TFib})$ is an algebraic weak factorization system.

Proof.
The classifier $t : 1 \to \Phi$ determines a fibered polynomial monad

$$P_t = \Phi_! t_* : \text{cSet} \to \text{cSet}$$

the algebras for which in $\text{cSet}/X$ are the trivial fibrations.
2. The fibration wfs \((\text{TCof}, \mathcal{F})\)

The \textbf{fibrations} \(\mathcal{F}\) are defined in terms of the trivial fibrations by

\[(f : F \to X) \in \mathcal{F} \quad \text{iff} \quad (\delta \Rightarrow f) \in \text{TFib}\]

where \(\delta \Rightarrow f\) is the \textbf{gap map} with \(\delta : 1 \to \mathbb{I}\) in \(\text{cSet}/\mathbb{I}\).

\[
\begin{array}{ccc}
F^{\mathbb{I}} & \longrightarrow & F \\
\downarrow \delta \Rightarrow f & & \downarrow \\
F & = & F \\
\downarrow & & \downarrow f \\
X^{\mathbb{I}} & \longrightarrow & X^{\mathbb{I}} \\
\end{array}
\]

The \textbf{trivial cofibrations} \(\text{TCof}\) are the maps that lift against \(\mathcal{F}\).

\[\text{TCof} := \mathfrak{M} \mathcal{F}\]
3. The weak equivalences $\mathcal{W}$

Let $\mathcal{W} := \mathrm{TFib} \circ \mathrm{TCoF}$.

**Proposition**

$(\mathcal{C}, \mathrm{TFib})$ and $(\mathrm{TCoF}, \mathcal{F})$ form a Barton premodel structure.

$$\mathrm{TCoF} = \mathcal{W} \cap \mathcal{C}$$

$$\mathrm{TFib} = \mathcal{W} \cap \mathcal{F}$$

**Corollary**

*If $\mathcal{W}$ satisfies 3-for-2, then $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a QMS.*
3. The weak equivalences $\mathcal{W}$

We use a universal fibration $\hat{U} \twoheadrightarrow U$ to show 3-for-2 for $\mathcal{W}$.

(i) there is a universal small map $\hat{V} \rightarrow V$
(ii) $U$ is the classifying type for fibration structures on $\hat{V} \rightarrow V$
(iii) $\hat{U} \rightarrow U$ is univalent
(iv) $U$ is fibrant
(v) fibrant $U$ implies 3-for-2 for $\mathcal{W}$

The idea of getting a QMS from univalence is due to Sattler.
3(i). The universal small map $\dot{V} \to V$

The **category of elements** functor $\int_C$

$$\int_C : \hat{C} \leftrightarrow \text{Cat} : \nu_C$$

always has a right adjoint **nerve** functor $\nu_C$.

**Proposition**

For any small map $Y \to X$ in $\hat{C}$ there is a canonical pullback

$$
\begin{array}{ccc}
Y & \to & \nu_C \: \dot{\text{set}}^{\text{op}} \\
\downarrow & & \downarrow \\
X & \to & \nu_C \: \text{set}^{\text{op}}
\end{array}
$$

since $\dot{\text{set}}^{\text{op}} \to \text{set}^{\text{op}}$ classifies small discrete fibrations in $\text{Cat}$. 
3(i). The universal small map $\dot{V} \to V$

The **category of elements** functor $\int_C$

\[ \int_C : \hat{C} \rightleftharpoons \text{Cat} : \nu_C \]

always has a right adjoint **nerve** functor $\nu_C$.

**Proposition**

*For any small map $Y \longrightarrow X$ in $\hat{C}$ there is a canonical pullback*

\[
\begin{array}{ccc}
Y & \longrightarrow & \nu_C \text{ set}^{\text{op}} \\
\downarrow & \quad & \downarrow \\
X & \longrightarrow & \nu_C \text{ set}^{\text{op}}
\end{array}
\]

\[\begin{array}{ccc}
& = & \\
\quad & \quad & \\
\dot{V} & \longrightarrow & V
\end{array}\]

since $\text{set}^{\text{op}} \to \text{set}^{\text{op}}$ classifies small **discrete fibrations** in $\text{Cat}$. 
3(ii). The universal fibration \( \hat{U} \to U \)

For any \( A \to X \) in cSet there is a **classifying type** \( \text{Fib}(A) \to X \), the sections of which correspond to fibration structures.

\[
\begin{array}{c}
\text{A} \\
\downarrow \\
\text{Fib}(A) \\
\downarrow \\
\text{X}
\end{array}
\]
3(ii). The universal fibration $\dot{U} \to U$

The construction of $\text{Fib}(A) \to X$ is stable under pullback.

\[
\begin{array}{ccc}
  f^*A & \to & A \\
  \downarrow & & \downarrow \\
  f^*\text{Fib}(A) & \to & \text{Fib}(A) \\
  \downarrow & & \downarrow \\
  Y & \to & X
\end{array}
\]

$f^*\text{Fib}(A) \cong \text{Fib}(f^*A)$

This uses the root functor $(-)^\Pi \dashv (-)_\Pi$. 
3(ii). The universal fibration $\dot{U} \to U$

Let $U$ be the type of fibration structures on $\dot{V} \to V$

\[
\begin{array}{c}
\dot{V} \\
\downarrow \\
U := \text{Fib}(\dot{V}) \longrightarrow V
\end{array}
\]

then define $\dot{U} \to U$ by pulling back.

\[
\begin{array}{c}
\dot{U} \\
\downarrow \\
\dot{V} \\
\downarrow \\
U \\
\downarrow \\
V
\end{array}
\]
3(ii). The universal fibration $\dot{U} \rightarrow U$

Since $\text{Fib}(\dot{-})$ is stable, the lower square is a pullback.
3(ii). The universal fibration $\dot{U} \rightarrow U$

Since $\text{Fib}(\rightarrow)$ is stable the lower square is also a pullback.

![Diagram](image)

But since $U = \text{Fib}(\dot{V})$ there is a section of $\text{Fib}(\dot{U})$. So $\dot{U} \rightarrow U$ is a fibration.
3(ii). The universal fibration $\hat{U} \rightarrow U$

A fibration structure $\alpha$ on a small map $A \rightarrow X$ determines a factorization $(a, \alpha)$ of its classifying map $a : X \rightarrow V$.

![Diagram]

- $A \rightarrow \hat{V}$
- $Fib(A) \rightarrow Fib(\hat{V})$
- $A \rightarrow X$
- $X \rightarrow V$
- $a$
- $(a, \alpha)$
- $\alpha$
3(ii). The universal fibration $\dot{U} \to U$

A fibration structure $\alpha$ on a small map $A \to X$ determines a factorization $(a, \alpha)$ of its classifying map $a : X \to V$,

\[
\begin{array}{ccc}
A & \to & \dot{V} \\
\downarrow \alpha & & \downarrow \dot{U} \\
X & \to & U \\
\end{array}
\]

which classifies $A \to X$ as a fibration since $\text{Fib}(\dot{V}) = U$. 
3(iii). \( \dot{U} \to U \) is univalent

The universal fibration \( \dot{U} \to U \) is \textbf{univalent} if the type

\[
\text{Eq}_B = \Sigma_B \text{Eq}(-, B) \to U
\]

of \textbf{based equivalences} is always a trivial fibration.

\[
\begin{array}{ccc}
C' & \rightarrow & \text{Eq}_B \\
\downarrow & & \downarrow \\
C & \rightarrow & U
\end{array}
\]

\( A \simeq B \)

Remark
In HoTT this implies \((A = B) \simeq (A \simeq B)\).
3(iii). $\dot{U} \to U$ is univalent

Unwinding (*) gives the **equivalence extension property**: weak equivalences extend along cofibrations $C' \hookrightarrow C$.
3(iii). $\hat{U} \rightarrow U$ is univalent

**Proposition**

*The universal fibration $\hat{U} \rightarrow U$ is univalent.*

Voevodsky proved this *classically* for Kan fibrations in $\text{sSet}$.

Coquand gave a constructive proof in *type theory* for $\text{cSet}$.

We have generalized Coquand’s proof to cartesian cubical sets.
3(iv). U is fibrant

Univalence of $\tilde{U} \to U$ implies that $U$ is fibrant.

**Proposition**

*The universe $\mathcal{U}$ is fibrant.*

Voevodsky proved this for Kan sSets using *minimal fibrations*. Shulman proved it using *3-for-2* for $\mathcal{W}$. Coquand proved it from univalence without 3-for-2 using *Kan composition* for cSets in type theory.

We give a general proof from univalence without using 3-for-2.
3(v). From fibrant U to 3-for-2

Finally, we can apply the following.

**Proposition (Sattler)**

$\mathcal{W}$ satisfies 3-for-2 if fibrations extend along trivial cofibrations.

\[ \begin{array}{ccc}
A & \to & A' \\
\downarrow & & \downarrow \\
X & \sim & X'
\end{array} \]

This is called the **fibration extension property**.
3. From fibrant $U$ to 3-for-2 for $\mathcal{W}$

Lemma
Given a universal fibration $\dot{U} \rightarrow U$ the FEP holds if $U$ is fibrant.
References

- M. Shulman, All $(\infty, 1)$-toposes have strict univalent universes, 2019.
Appendix: U is fibrant (sketch)

It suffices to show the following.

Proposition

Evaluation at the generic point $U^\square \to U$ is a trivial fibration.

Proof.
We need a diagonal filler for any cofibration $c$.

\[
\begin{array}{ccc}
C' & \xrightarrow{a} & U^\square \\
\downarrow{c} & & \downarrow{\delta} \\
C & \xrightarrow{b} & U
\end{array}
\]
Transposing by $\mathbb{I}$ and using the classifying property of $U$ gives the following equivalent problem.
Appendix: U is fibrant (sketch)

Apply the functor \((-\)) \times \mathbb{I} to the left face to get:

\[
\begin{align*}
A_0 & \quad \longrightarrow \quad A \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \\
C' & \quad \longrightarrow \quad C' \times \mathbb{I} \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \\
B & \quad \longrightarrow \quad D \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \\
C & \quad \longrightarrow \quad C \times \mathbb{I} \\
\end{align*}
\]
Appendix: U is fibrant (sketch)

Apply the functor \((-\) \times \mathbb{I}\) to the left face to get:

\[
\begin{array}{c}
A_0 \\
\downarrow \\
C' \\
\downarrow \\
B \\
\downarrow \\
C
\end{array} \quad \begin{array}{c}
A \\
\downarrow \\
C' \times \mathbb{I} \\
\downarrow \\
D \\
\downarrow \\
C \times \mathbb{I}
\end{array} \quad \begin{array}{c}
A_0 \times \mathbb{I} \\
\downarrow \\
B \times \mathbb{I}
\end{array}
\]

There is a weak equivalence \(e : A \sim A_0 \times \mathbb{I}\) to which we can apply the EEP.
Appendix: U is fibrant (sketch)

Apply the functor $(-) \times \mathbb{I}$ to the left face to get:

There is a weak equivalence $e : A \simeq A_0 \times \mathbb{I}$ to which we can apply the EEP. This produces the required fibration $D \rightarrow Z \times \mathbb{I}$. \qed