

Path Types in Algebraic Type Theory

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Outline

- 1) Review of Natural Models
- 2) Path Types
- 3) Examples
- 4) Cubical Kan Structure

1. Natural Models

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Def. A natural model consists of:

- a cat $\mathcal{C}$ of contexts $\sigma: \Delta \rightarrow \mathbb{T}$,
- presheaves $\mathbb{T}, \tilde{\mathbb{T}}$ of types & terms,
- a natural transformation $t: \tilde{\mathbb{T}} \rightarrow \mathbb{T}$ that types the terms,

type
theory

$$\mathbb{T} \vdash a : A$$



$$\begin{array}{ccc} & & \tilde{\mathbb{T}} \\ & \nearrow a & \downarrow t \\ y^{\mathbb{T}} & \xrightarrow{A} & \mathbb{T} \end{array}$$

$\hat{\mathcal{C}}$

- $t: \tilde{T} \longrightarrow T$ is representable:

$$\begin{array}{ccc}
 \textcolor{blue}{y} \vdash T.A & \xrightarrow{\quad} & \tilde{T} \\
 \downarrow & \perp & \downarrow t \\
 \textcolor{blue}{y} \vdash T & \xrightarrow{\quad A \quad} & T
 \end{array}$$

this models context extension,

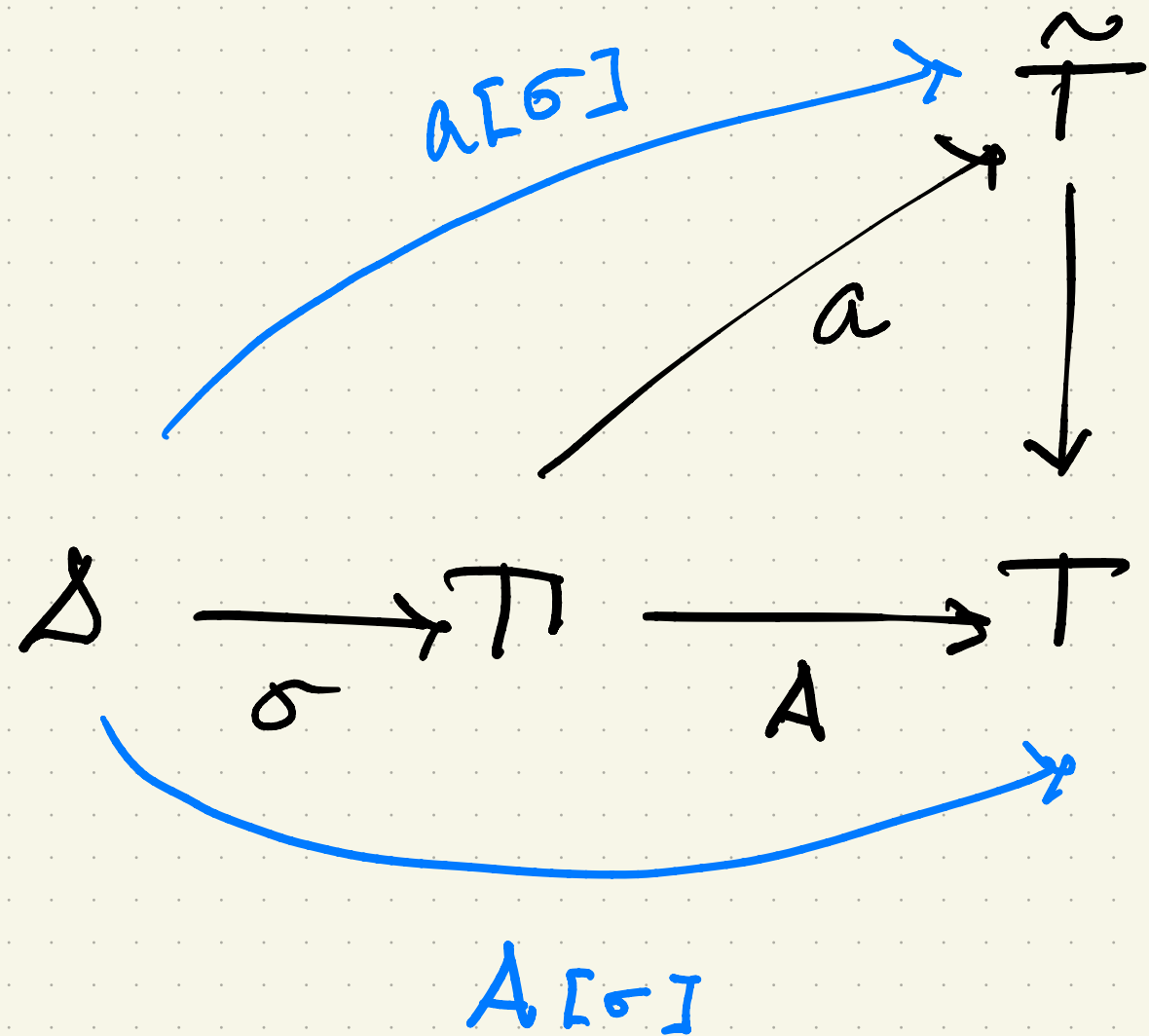
$$\frac{T \vdash A}{T.A \vdash}$$

$$\pi_A: T.A \longrightarrow T$$

Substitution is modelled by composition*.

$$\frac{\Pi \vdash a : A \quad \sigma : \Delta \rightarrow \Pi}{\Delta \vdash a[\sigma] : A[\sigma]}$$

$$\Delta \vdash a[\sigma] : A[\sigma]$$



*) not pullback!

Type formers Σ, Π are modelled using the
polynomial endofunctor of $t: \tilde{T} \rightarrow T$:

$$P_t: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$$

$$P_t(X) = \sum_{A:T} X^A$$

$$\begin{array}{ccccc}
 & \tilde{T}^* & \nearrow & \hat{\mathcal{C}}/\tilde{T} & \xrightarrow{t_*} & \hat{\mathcal{C}}/T & \searrow & T! \\
 & & & & & & & \downarrow \\
 \hat{\mathcal{C}} & \xrightarrow{\quad P_t \quad} & \hat{\mathcal{C}} & & & & & \mathcal{C}
 \end{array}$$

E.g. Π -Rules of Form - Intro - Elim - Comp :

$$\begin{array}{ccc}
 P_t \tilde{T} & \xrightarrow{\lambda} & \tilde{T} \\
 P_t t \downarrow & & \downarrow t \\
 P_t T & \xrightarrow{\pi} & T
 \end{array}$$

$\Leftrightarrow$

$$\text{Form} \quad \frac{\Pi, A \vdash B}{\Pi \vdash \Pi_A B}$$

$$\frac{\Pi, A \vdash b : B}{\Pi \vdash \lambda_A b : \Pi_A B} \quad \text{Intro}$$

The Σ -Rules state a multiplication

$$(\Sigma, \text{pair}) : P_t \circ P_t \Rightarrow P_t$$

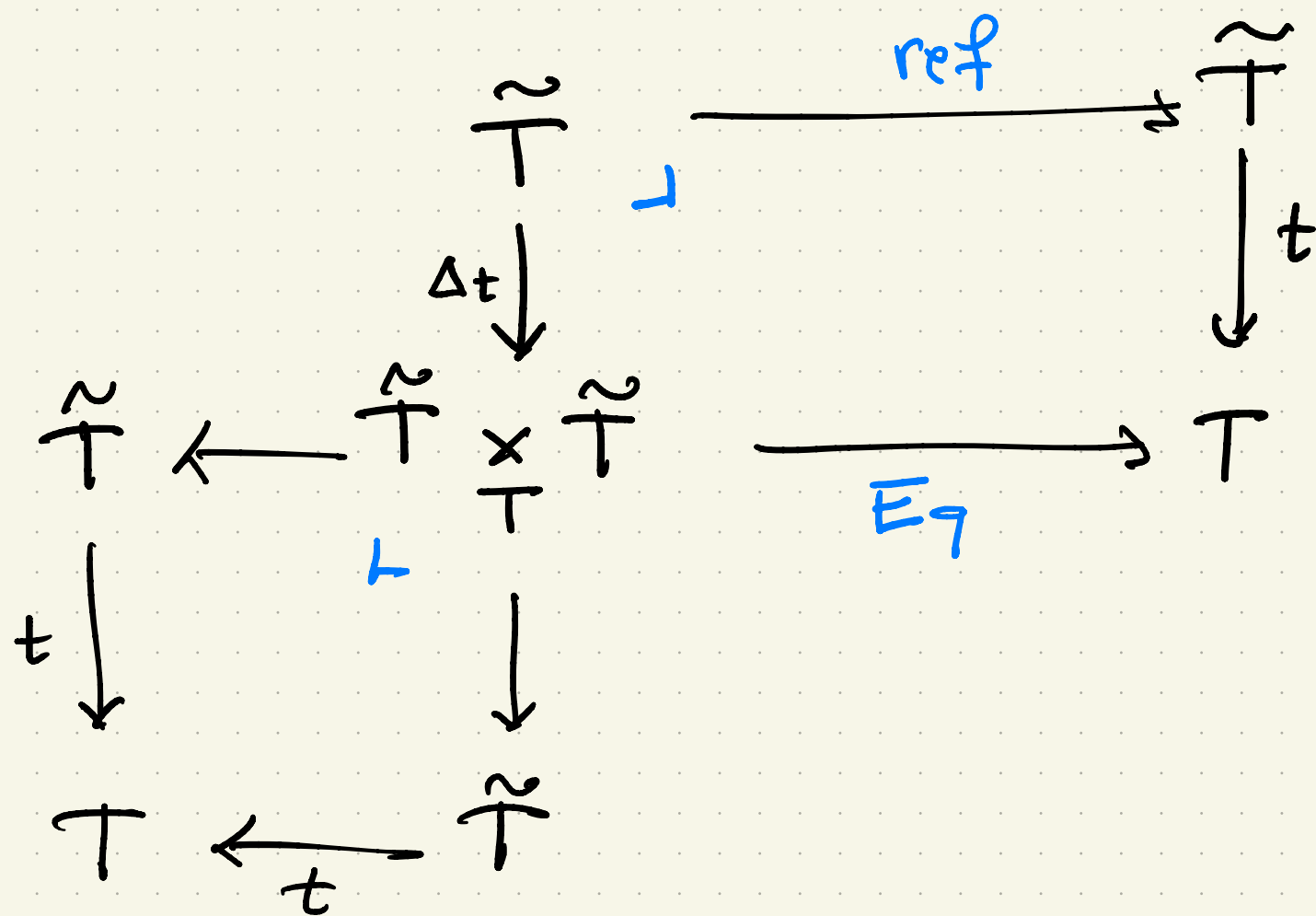
in the cat $\text{Poly}(\hat{\mathcal{C}})$ of polynomial functors,

$$\begin{array}{ccc}
 Q & \xrightarrow[\text{pair}]{\perp} & \tilde{T} \\
 \downarrow t \cdot t & & \downarrow t \\
 P_t T & \xrightarrow[\Sigma]{} & T
 \end{array}$$

where

$$P_{t \cdot t} = P_t \circ P_t .$$

The model $t: \tilde{T} \rightarrow T$ has extensional identity types just if there's a pullback:



$$\frac{\Pi \vdash a:A \quad \Pi \vdash b:A}{\Pi \vdash Eq_A(a,b)}$$

$$\frac{\Pi \vdash a:A}{\Pi \vdash ref(a): Eq_A(a,a)}$$

2. Path Types

(8)

Now fix an interval in $\hat{\mathcal{C}}$,

$$1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} I.$$

For each object X we have a pathobject,

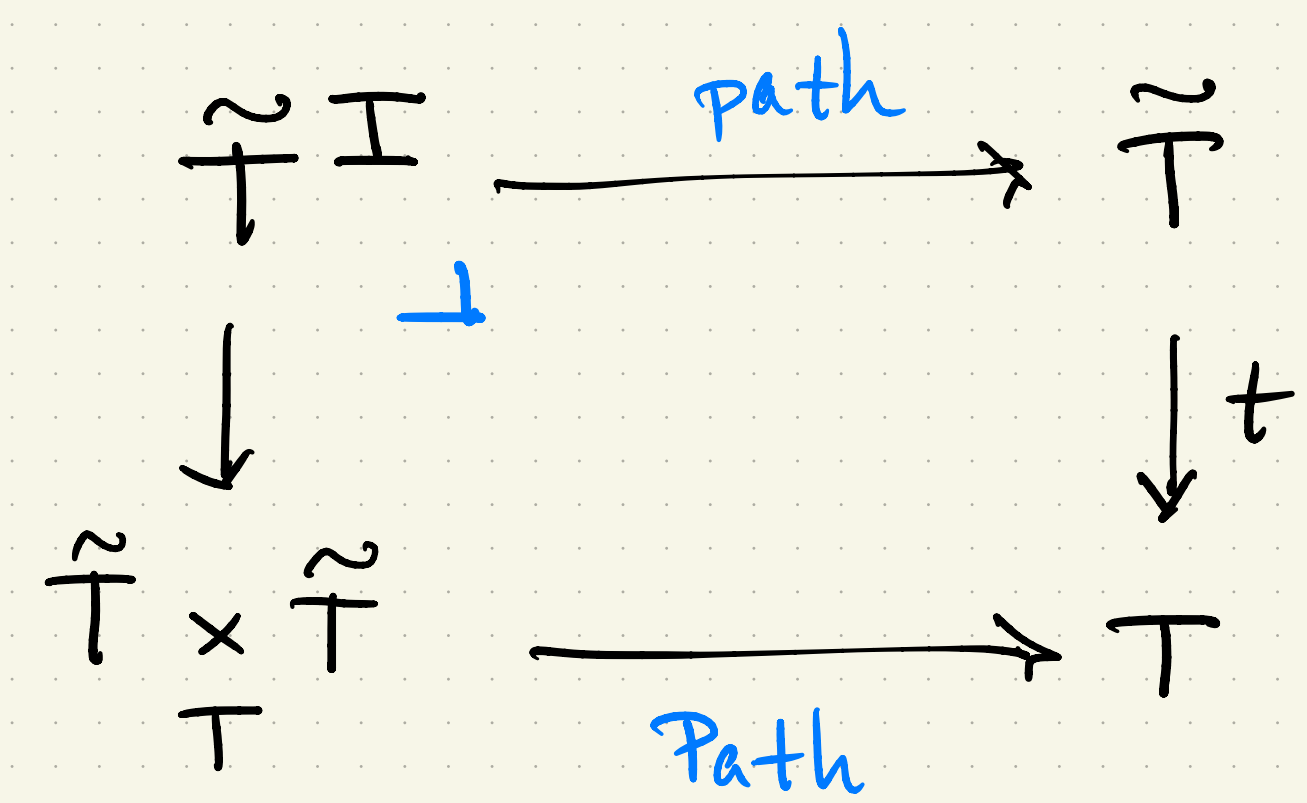
$$X \longrightarrow X^I \rightrightarrows X$$

and a factorization
of Δ_X .

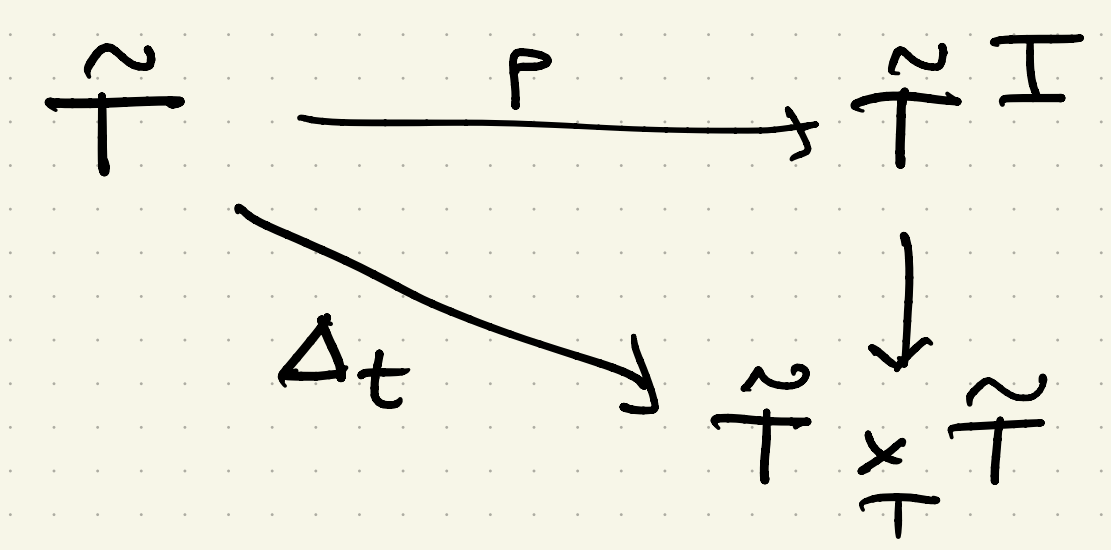
$$\begin{array}{ccc} X & \xrightarrow{p} & X^I \\ & \searrow \Delta_X & \downarrow \\ & & X \times X \end{array}.$$

Def. $t: \tilde{T} \rightarrow T$ has path types if

there are maps:



where

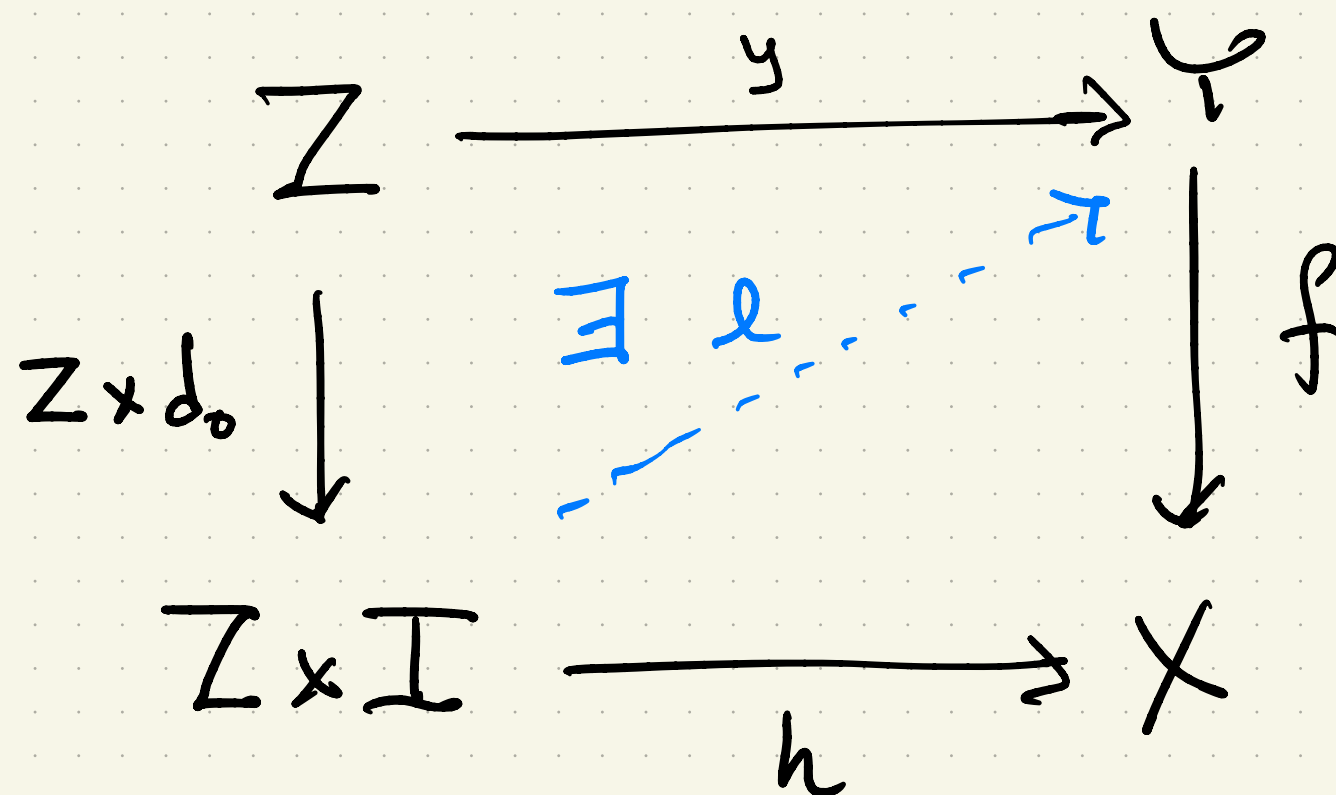


is the path object factorization of Δ_t over T .

We also need the following:

Def. $f: Y \rightarrow X$ is a Hurewicz fibration if

for all Z & y & h ,

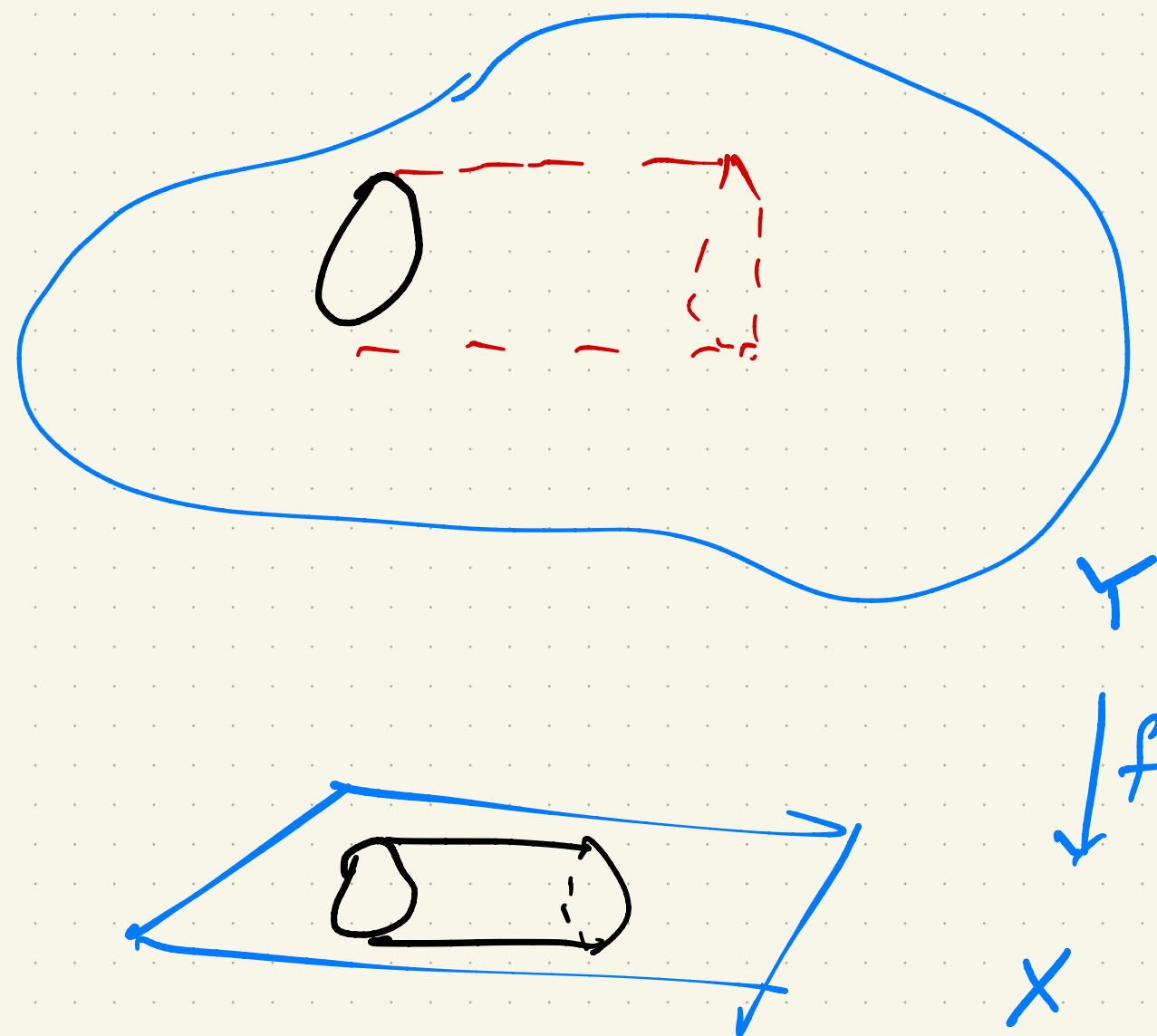
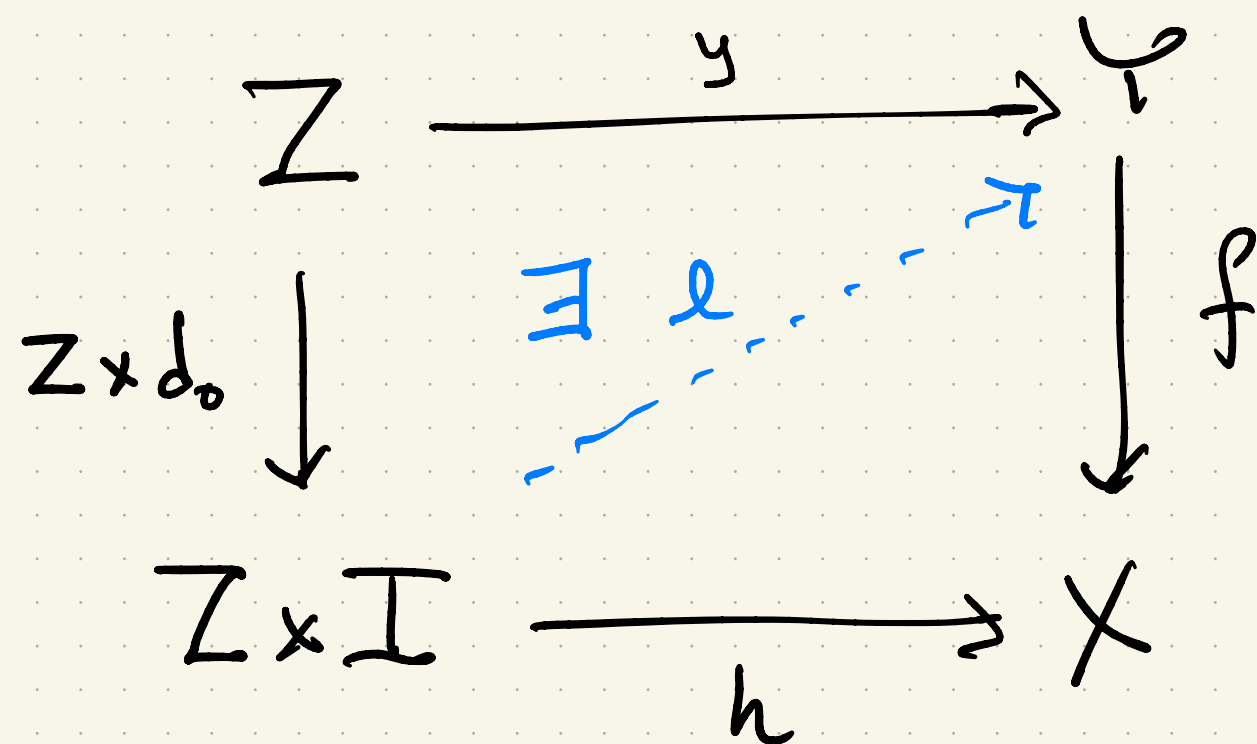


"Homotopy
lifting
property"

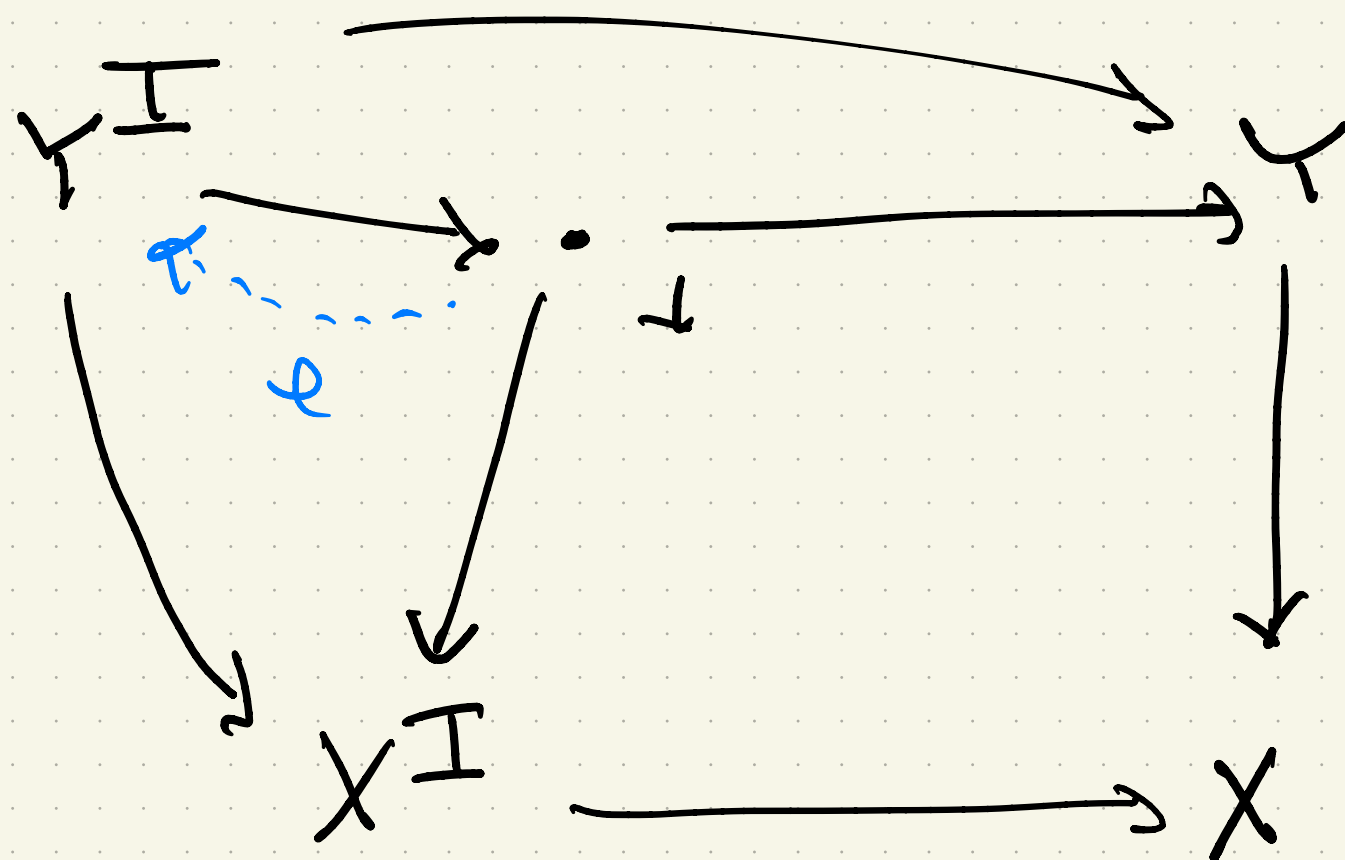
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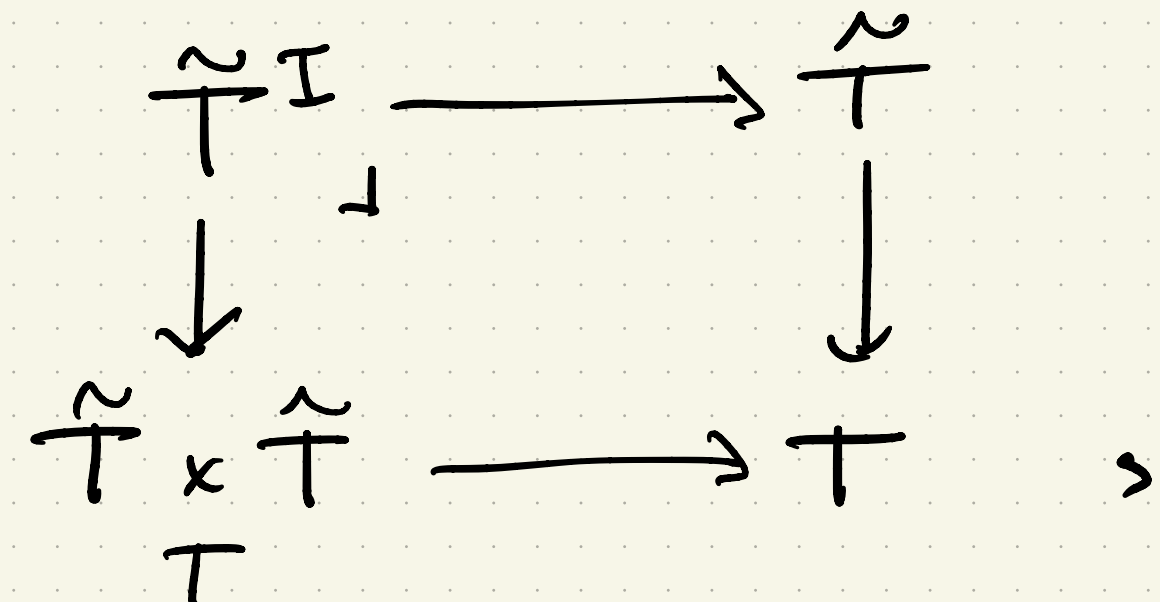
$f: Y \rightarrow X$ is Hurwicz if there's a **lifting** l :



Call f normal if $l(f, -) = p$.

Prop. Suppose $t: \tilde{T} \rightarrow T$,

i) has path types



ii) is a normal Hurewicz fibration.

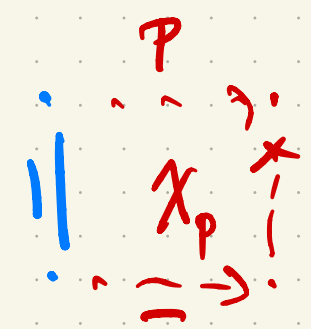
Then t models intensional Id-types.

Lemma. If $t: \tilde{T} \rightarrow T$ satisfies (i) & (ii),

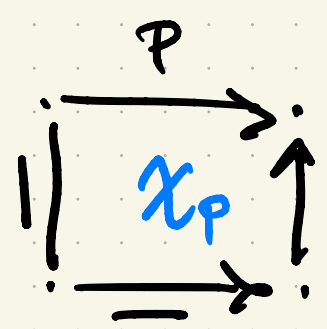
then any type $A \rightarrow X$ classified by t has a connection $\chi: A^I \rightarrow A^{I \times I}$.

pf.

$$A^I \xrightarrow{\chi} A^{I \times I}$$



$$\cdot \xrightarrow{p} \cdot$$



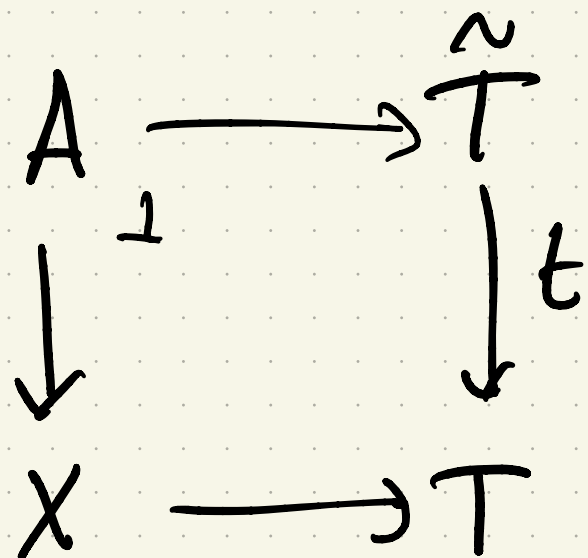
$$\begin{array}{ccc} \cdot & \xrightarrow{p} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

$$\begin{array}{c} A^I \\ \downarrow \\ A \times A \end{array}$$

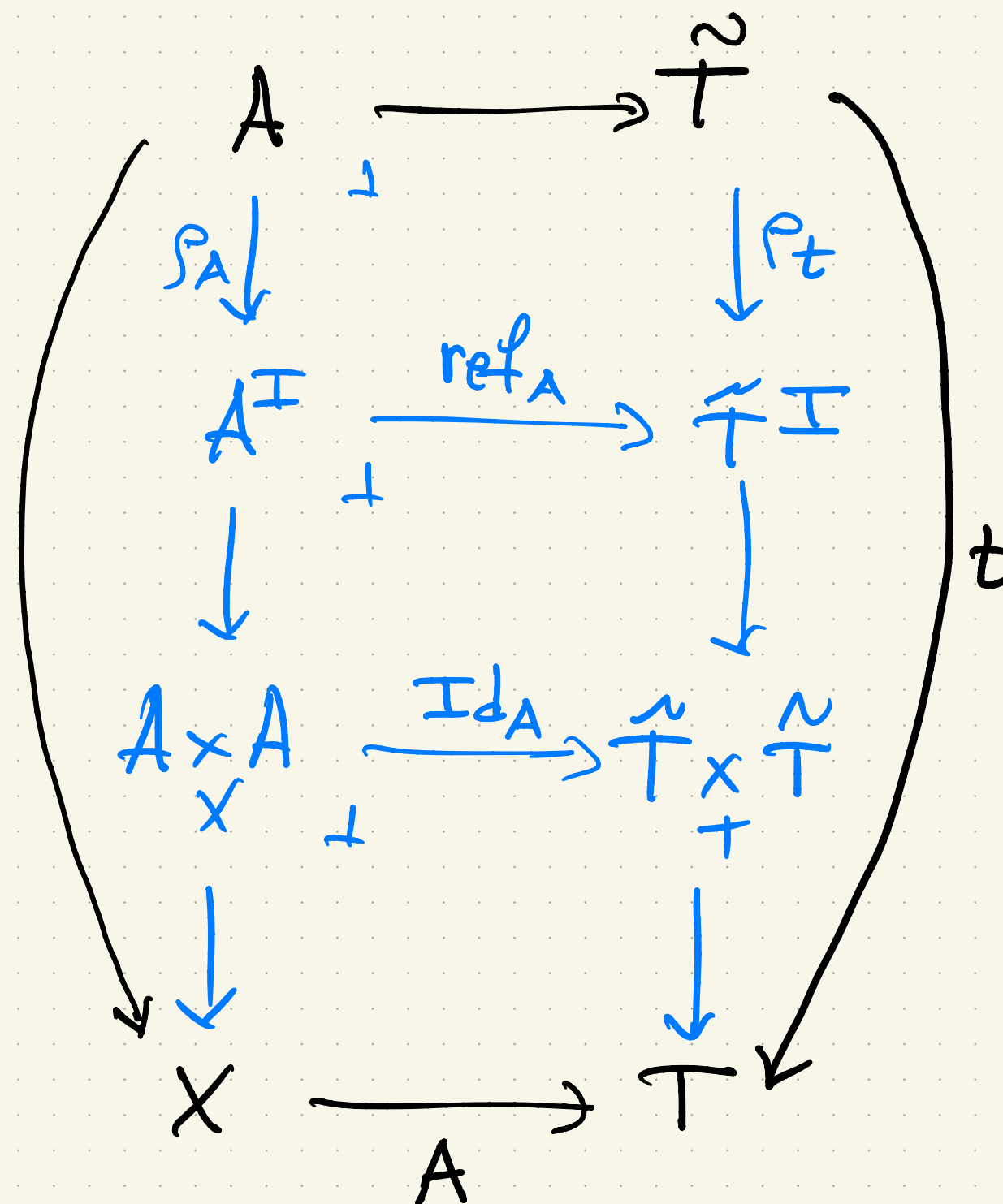
Pf. of Prop

(15)

1) Given:



2) We have:



3) Models:

$$X \vdash A$$

Form

$$\frac{}{X, x, y:A \vdash \text{Id}_A(x, y)}$$

•/.

Intro

$$\frac{}{X, x:A \vdash \text{ref}_A(x) : \text{Id}_A(x, x)}$$

4) NTS:

$$x, y: A, p: \text{Id}_A(x, y) \vdash C$$

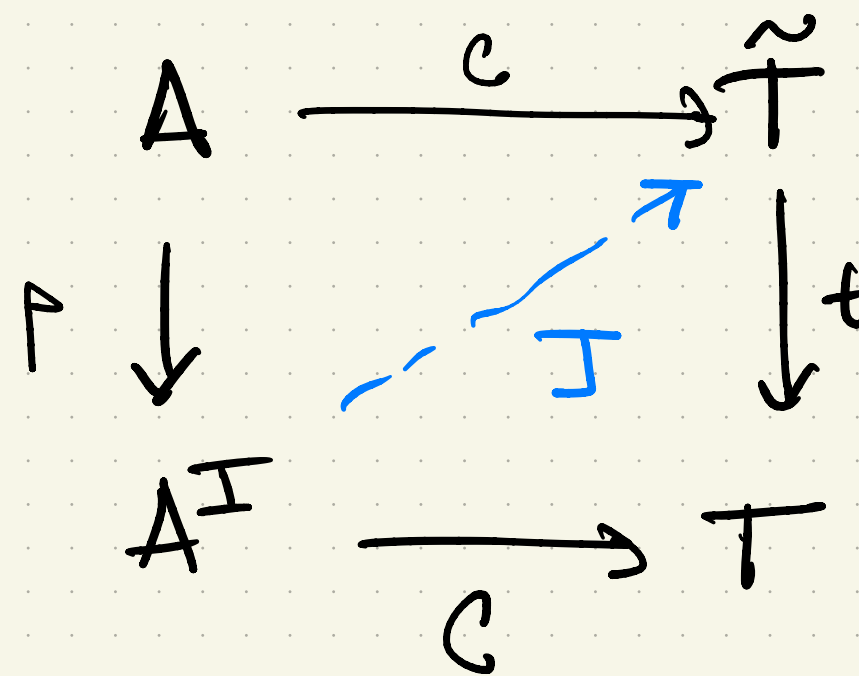
$$x: A \vdash c: C_p$$

Elim

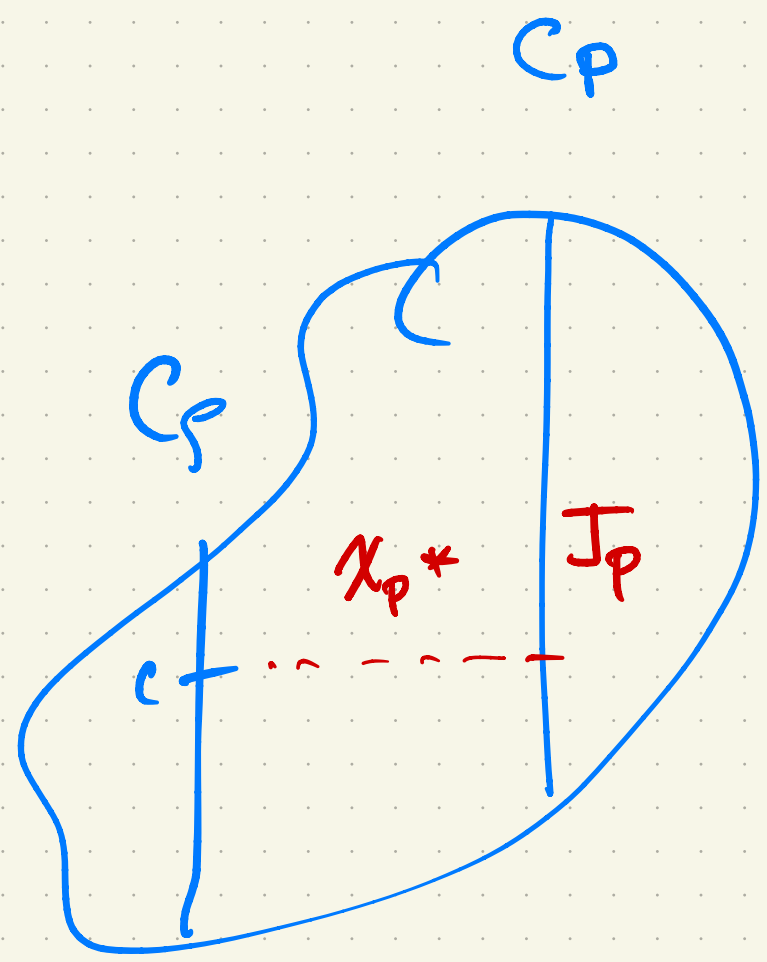
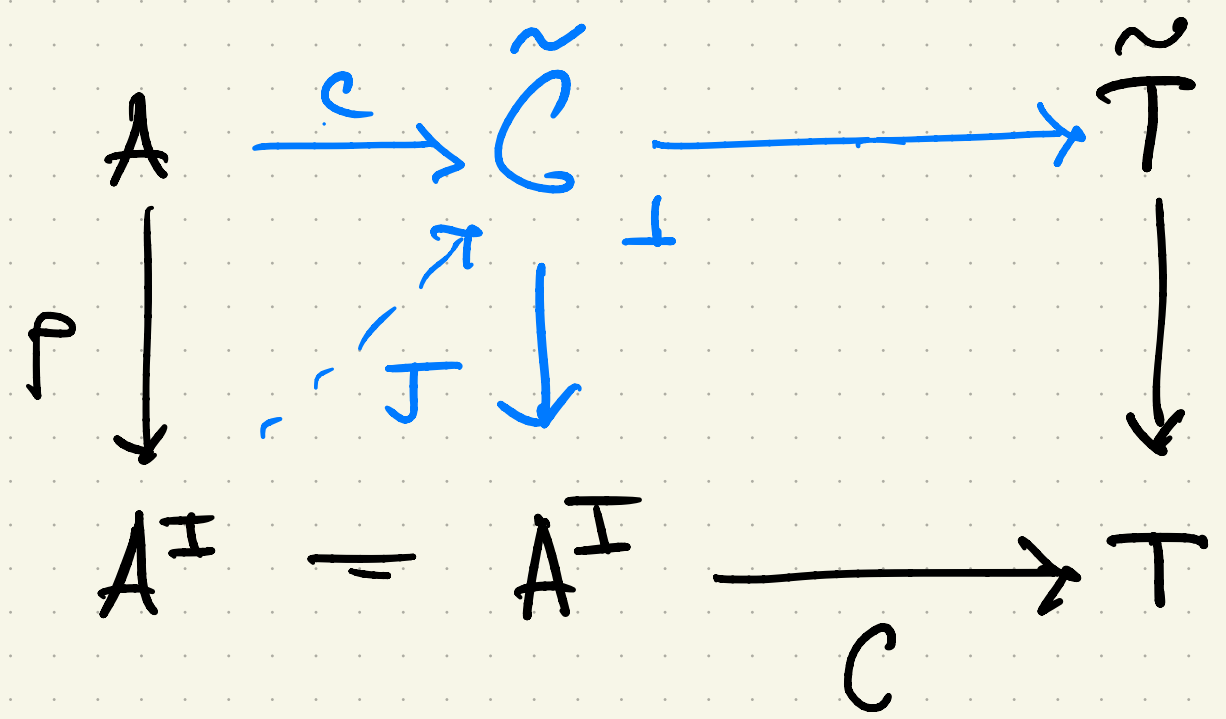
$$x, y: A, p: \text{Id}_A(x, y) \vdash J: C$$

Comp

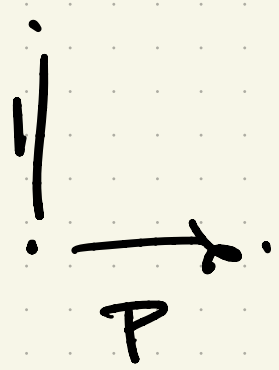
$$\frac{\cdot}{x: A \vdash J_p \equiv c}$$



Equivalently :

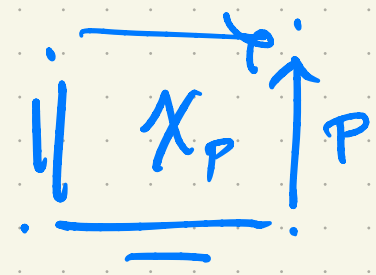


5) Take and get



$c: C_p$

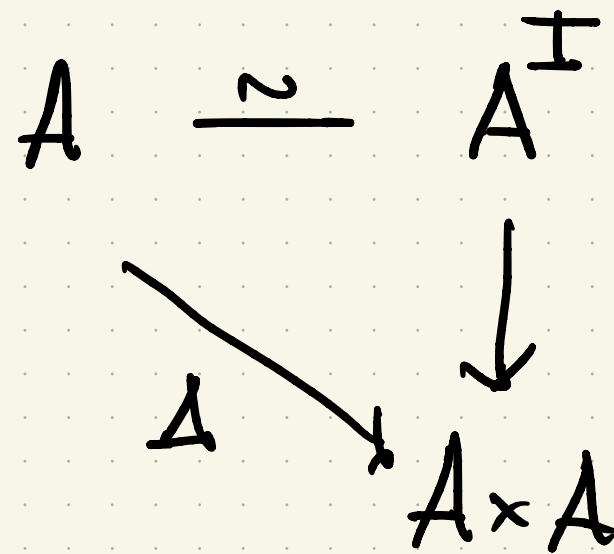
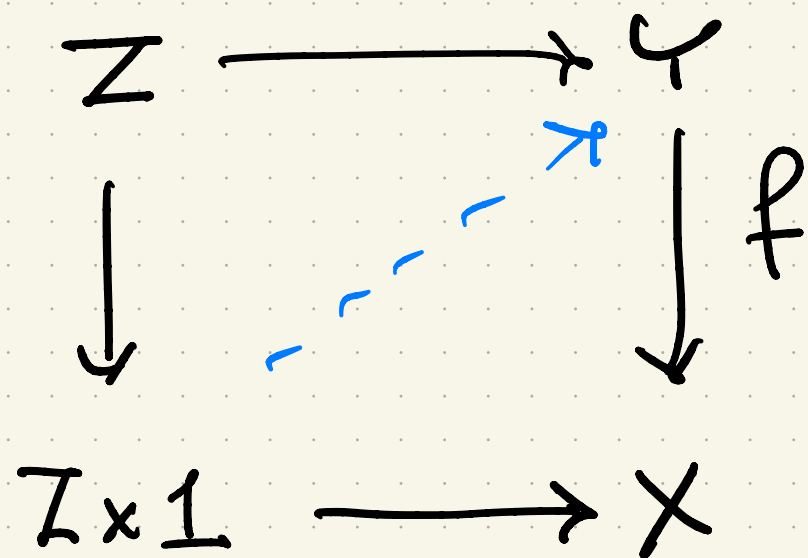
$$\chi_p + c =: J_p$$



3. Examples

L18

1) $I = \mathbb{1}$ in any LCC:

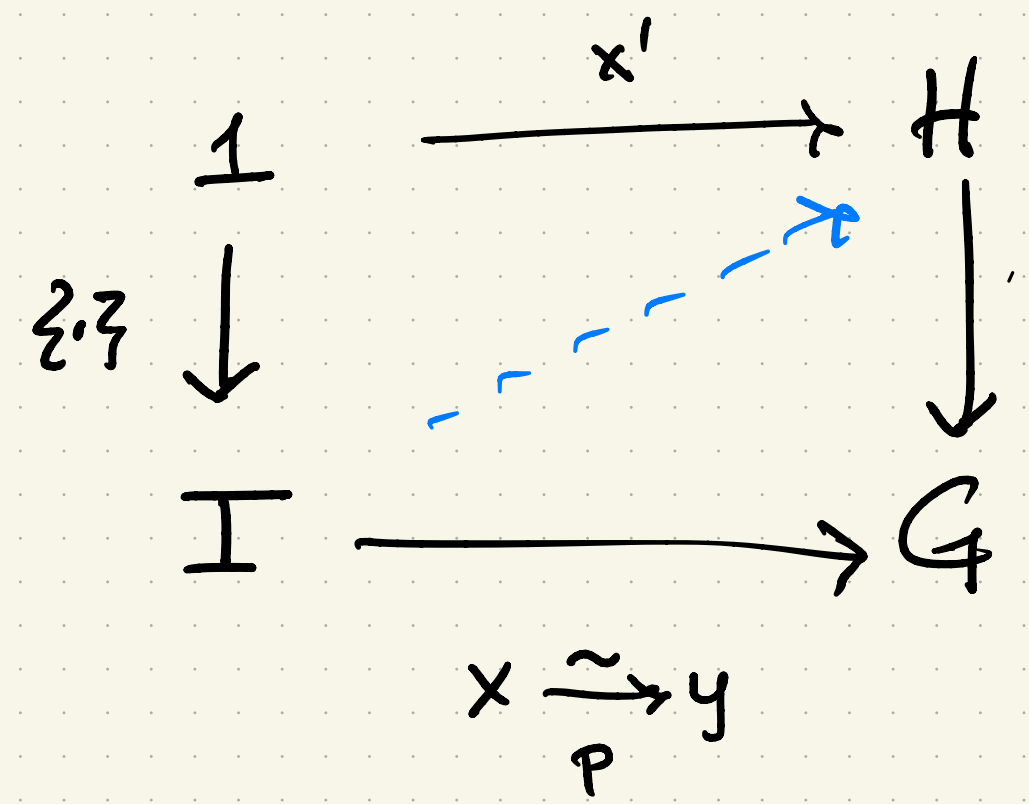


all $f: Y \rightarrow X$
are H-fibrations

path types
are diagonals

Extensional TT

2) $I = \cdot \xrightarrow{\sim} \cdot$ in $GpdS$:



$$G^I = G^{\downarrow}$$

$$\downarrow \qquad \downarrow \text{ (dom, cod) }$$

$$G \times G = G \times G$$

Hurewicz
 = (iso-)fibration

Path object
 = arrow groupoid

Usual H-S Groupoid model

3) $I = [1]$ in $s\text{Set} = \text{Set}^{\Delta^{\text{op}}}$:

Kan fibrations are Hurewicz,
so Id.types follow from the Prop.

4) $I = [1]$ in $c\text{Set} = \text{Set}^{\#^{\text{op}}}$:

Cubical Kan fibrations (CT 2023).

New: Path types $\Rightarrow$ n-box filling f. all $n \geq 0$!

4. Box Filling

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Def. $f: Y \rightarrow X$ has n-box filling if for all open boxes $i^n: U \hookrightarrow I^n$, and all b, c , there's a diagonal filler φ , as in:

$$\begin{array}{ccc} U & \xrightarrow{b} & Y \\ i^n \downarrow & \nearrow \varphi & \downarrow f \\ I^n & \xrightarrow{c} & X \end{array}$$

Briefly,

$$i^n \dashv f$$

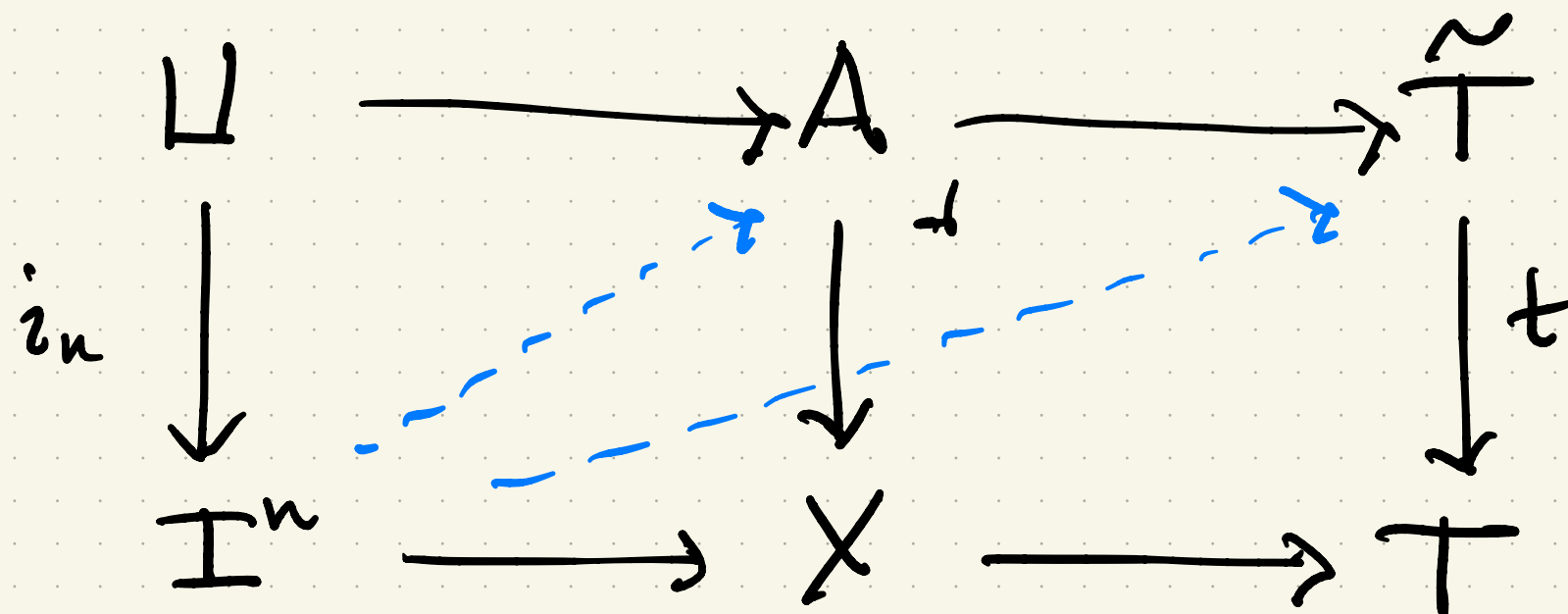
Prop. Suppose the model $t: \tilde{T} \rightarrow T$

(i) has path types ,

(ii) is Hurewicz .

Then any type family $A \rightarrow X$ classified by t
has n -box filling f. all $n \geq 0$.

Pf.



STS:

$$i_n \vdash t$$

$$\forall n \geq 0 .$$

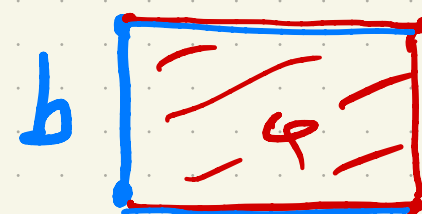
LemmaFor any $A \rightarrow X$,

$$i_n \vdash A^I \iff i_{n+1} \vdash A$$

Pf.

$$\begin{array}{ccc}
 1 & \xrightarrow{b} & A^I \\
 d_0 \downarrow & \nearrow \varphi & \downarrow \\
 I & \xrightarrow{c} & A \times_X A
 \end{array}$$

$$i_1 \vdash A^I$$



$$\begin{array}{c}
 \xrightarrow{c} \\
 \xrightarrow{c'}
 \end{array}$$

$$i_2 \vdash A$$

Pf. of Prop.

1) t Hurewicz $\Rightarrow$

$$\begin{array}{ccc} i_1 \downarrow & \xrightarrow{\quad} & \tilde{T}^1 \\ & \nearrow & \downarrow t \\ I & \xrightarrow{\quad} & T \end{array}$$

$i_1 \pitchfork t$,

2) path types $\Rightarrow$

$$\begin{array}{ccc} \tilde{T}^I & \xrightarrow{\quad} & \hat{T} \\ t^I \downarrow & \lrcorner & \downarrow t \\ \tilde{T} \times \hat{T} & \xrightarrow{\quad} & T \end{array}$$

$i_1 \pitchfork t^I$,

3) Lemma $\Rightarrow i_2 \pitchfork t$,

Etc., by induction on $n > 0$.

THANKS!



Reference:

S. A. & J. Hua: Path Types in ATT,
arXiv, 2026.