Path Types

in

Algebraic Type Theory

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Outline

1) Review of Natural Models

2) Path Types

3) Examples

4) Cubical Kan Structure

1. Natural Models

Def. A natural model consists of:

- a cat $\mathcal{C}$ of contexts $\sigma : \Delta \to \Gamma$,
- presheaves $T, \tilde{T}$ of types & terms,
- a natural transformation $t : \tilde{T} \to T$ that types the terms,

type theory

$$\Gamma \vdash a : A \quad \Leftrightarrow \quad \begin{tikzcd} & \tilde{T} \arrow[d, "t"] \\ y\Gamma \arrow[ur, "a"] \arrow[r, "A"'] & T \end{tikzcd}$$

$\widehat{\mathcal{C}}$

- $t : \tilde{T} \longrightarrow T$ is <u>representable</u> :

$$\begin{tikzcd} \Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\ \Gamma \arrow[r, "A"'] & T \end{tikzcd}$$

this models context extension,

$$\frac{\Gamma \vdash A}{\Gamma.A \vdash} \qquad \pi_A : \Gamma.A \longrightarrow \Gamma \, .$$

- Substitution is modelled by composition*.

$$\frac{\Pi \vdash a : A \qquad \sigma : \Delta \to \Pi}{\Delta \vdash a[\sigma] : A[\sigma]}$$

$$\begin{tikzcd}
 & & \tilde{T} \arrow[d] \\
\Delta \arrow[r, "\sigma"'] \arrow[urr, bend left, "{a[\sigma]}"] \arrow[rr, bend right, "{A[\sigma]}"'] & \Pi \arrow[r, "A"'] \arrow[ur, "a"] & T
\end{tikzcd}$$

*) not pullback!

- Type formers Σ, Π are modelled using the polynomial endofunctor of $t : \tilde{T} \to T$:

$$P_t : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}$$

$$P_t(X) = \sum_{A:T} X^A$$

$$\begin{tikzcd}
& \hat{\mathcal{C}}/\tilde{T} \arrow[r, "t_*"] & \hat{\mathcal{C}}/T \arrow[dr, "T!"] & \\
\hat{\mathcal{C}} \arrow[ur, "\tilde{T}^*"] \arrow[rrr, dashed, "P_t"] & & & \mathcal{C}
\end{tikzcd}$$

.

- E.g. Π-Rules of Form - Intro - Elim - Comp

$$\Longleftrightarrow \qquad \begin{tikzcd} P_t \tilde{T} \arrow[r, "\lambda"] \arrow[d, "P_t t"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\ P_t T \arrow[r, "\pi"'] & T \end{tikzcd} .$$

$$\text{Form}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash \Pi_A B} \qquad\qquad \frac{\Gamma, A \vdash b : B}{\Gamma \vdash \lambda_A . b : \Pi_A B}\ \text{Intro}$$

- The Σ-Rules state a multiplication

$$(\Sigma, \mathrm{pair}) : P_t \circ P_t \Longrightarrow P_t$$

in the cat $\mathrm{Poly}(\hat{\mathcal{E}})$ of polynomial functors,

$$\begin{tikzcd}
Q \arrow[r, "\mathrm{pair}"'] \arrow[d, "t\cdot t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\
P_t T \arrow[r, "\Sigma"'] & T
\end{tikzcd}$$

where

$$P_{t\cdot t} = P_t \circ P_t .$$

- The model $t : \tilde{T} \longrightarrow T$ has extensional identity types just if there's a pullback:

$$
\begin{tikzcd}
& \tilde{T} \arrow[r, "\mathrm{ref}"] \arrow[d, "\Delta_t"'] \arrow[dr, phantom, "\lrcorner" very near start] & \tilde{T} \arrow[d, "t"] \\
\tilde{T} \arrow[d, "t"'] & \tilde{T} \times_T \tilde{T} \arrow[l] \arrow[r, "\mathrm{Eq}"'] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] & T \\
T & \tilde{T} \arrow[l, "t"] &
\end{tikzcd}
$$

$$
\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathrm{Eq}_A(a,b)}
$$

$$
\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{ref}(a) : \mathrm{Eq}_A(a,a)}
$$

2. Path Types

Now fix an interval in $\hat{\mathcal{C}}$,

$$1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} I \,.$$

For each object X we have a pathobject,

$$X \longrightarrow X^I \rightrightarrows X$$

and a factorization of Δ_X.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "\Delta_X"'] & X^I \arrow[d] \\ & X \times X \end{tikzcd} .$$

<u>Def.</u> $t : \tilde{T} \longrightarrow T$ has <u>path types</u> if

there are maps:

$$\begin{tikzcd}
\tilde{T}^I \arrow[r, "\text{path}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\
\tilde{T} \times_T \tilde{T} \arrow[r, "\text{Path}"'] & T
\end{tikzcd}$$

where

$$\begin{tikzcd}
\tilde{T} \arrow[r, "p"] \arrow[dr, "\Delta_t"'] & \tilde{T}^I \arrow[d] \\
& \tilde{T} \times_T \tilde{T}
\end{tikzcd}$$

is the <u>pathobject factorization</u> of Δ_t over T.

We shall also need the following:

<u>Def.</u> $f: Y \to X$ is a <u>Hurewicz fibration</u> if for all Z & y & h,

$$\begin{tikzcd} Z \arrow[r, "y"] \arrow[d, "Z \times d_0"'] & Y \arrow[d, "f"] \\ Z \times I \arrow[r, "h"'] \arrow[ur, dashed, "\exists\, \ell"] & X \end{tikzcd}$$

"Homotopy lifting property"

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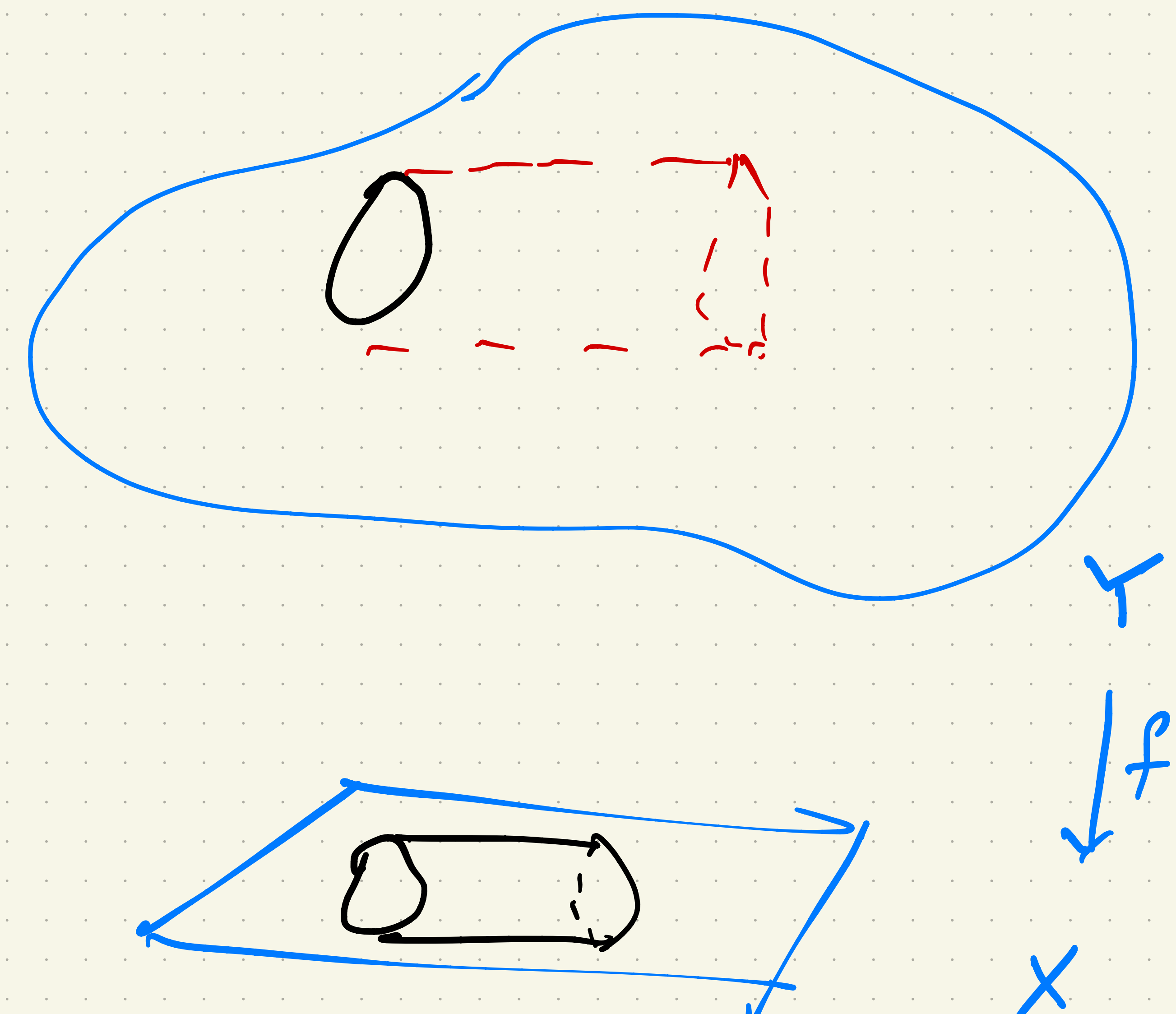


<u>Fact</u> If $f : Y \longrightarrow X$ is Hurewicz, then there's a section $\ell : X^I \times_X Y \longrightarrow Y^I$:

$$\begin{tikzcd}
Y^I \arrow[rrd, bend left] \arrow[dr] \arrow[ddr, bend right] & & \\
& \bullet \arrow[r] \arrow[d] \arrow[ul, dashed, bend left, "\ell"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
& X^I \arrow[r] & X
\end{tikzcd}$$

Call f <u>normal</u> if $\ell(p, -) = p$.

Prop. Suppose $t : \tilde{T} \to T$,

i) has path types
$$\begin{tikzcd} \tilde{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d] \\ \tilde{T} \times_T \tilde{T} \arrow[r] & T \end{tikzcd},$$

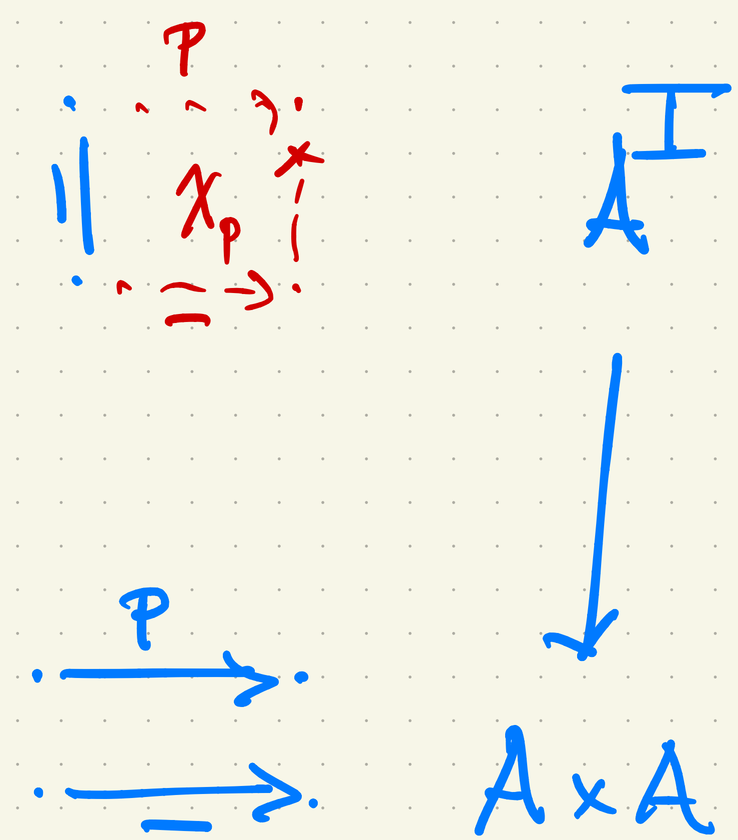
ii) is a normal Hurewicz fibration.

Then t models intensional Id-types.

Lemma. If $t:\tilde{T}\to T$ satisfies (i) & (ii), then any type $A\to X$ classified by t has a connection $\chi : A^{I} \longrightarrow A^{I\times I}$.

Pf.

$$A^{I} \xrightarrow{\ \chi\ } A^{I\times I}$$



Pf. of Prop

1) Given:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & \tilde{T} \arrow[d, "t"] \\ X \arrow[r] & T \end{tikzcd}$$

2) We have:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\rho_A"'] \arrow[ddd, bend right] \arrow[dr, phantom, "\perp"] & \tilde{T} \arrow[d, "\rho_t"] \arrow[ddd, bend left, "t"] \\ A^I \arrow[r, "\mathrm{ref}_A"] \arrow[d] \arrow[dr, phantom, "\perp"] & \tilde{T}^I \arrow[d] \\ A \times_X A \arrow[r, "\mathrm{Id}_A"] \arrow[d] \arrow[dr, phantom, "\perp"] & \tilde{T} \times_T \tilde{T} \arrow[d] \\ X \arrow[r, "A"'] & T \end{tikzcd}$$

3) Models:

$$\text{Form}\ \frac{X \vdash A}{X,\ x, y : A \vdash \mathrm{Id}_A(x,y)}$$

$$\text{Intro}\ \frac{\%}{X,\ x : A \vdash \mathrm{ref}_A(x) : \mathrm{Id}_A(x,x)}$$

4) NTS:

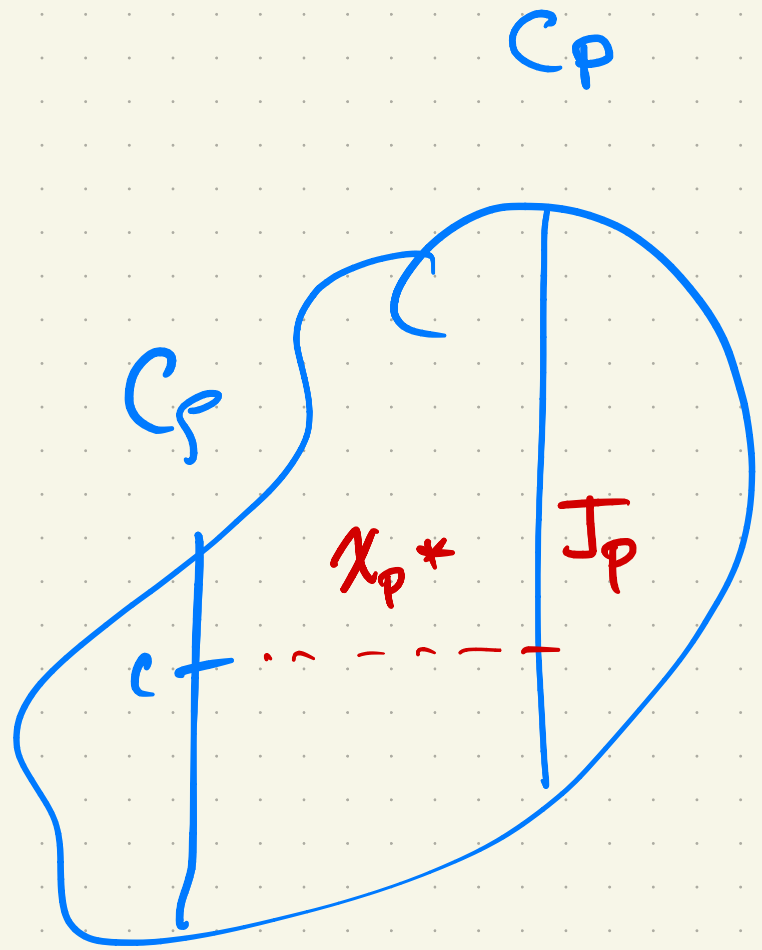
$$\text{Elim}\ \frac{x,y:A,\ p:\mathrm{Id}_A(x,y)\vdash C \qquad x:A\vdash c:C_\rho}{x,y:A,\ p:\mathrm{Id}_A(x,y)\vdash J:C}$$

$$\text{Comp}\ \frac{\%}{x:A\vdash J\rho\equiv c}$$

$$\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, "p"'] & \tilde{T} \arrow[d, "t"] \\
A^I \arrow[r, "C"'] \arrow[ur, dashed, "J"'] & T
\end{tikzcd}$$

Equivalently :

$$\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, "p"'] & \tilde{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d] \\
A^I \arrow[r, equal] \arrow[ur, dashed, "J"] & A^I \arrow[r, "c"'] & T
\end{tikzcd}$$



$\chi_p * c =: J_p$

5) Take
$$\begin{tikzcd}
\cdot \arrow[d, equal] & \\
\cdot \arrow[r, "p"'] & \cdot
\end{tikzcd}$$
and $c : C_p$ get

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\chi_p"] & \cdot \\
\cdot \arrow[r, equal] & \cdot \arrow[u, "p"']
\end{tikzcd}$$

3. Examples

1) $I = 1$ in any LCC:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ I \times 1 \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[dr, "\Delta"'] & A^I \arrow[d] \\ & A \times A \end{tikzcd}$$

all $f : Y \to X$ are H-fibrations

path types are diagonals

Extensional TT.

2) $I = \bullet \xrightarrow{\sim} \bullet$ in Gpds :

$$\begin{tikzcd} 1 \arrow[r, "x'"] \arrow[d, "\{i\}"'] & H \arrow[d] \\ I \arrow[r, "x \xrightarrow[p]{\sim} y"'] \arrow[ur, dashed] & G \end{tikzcd}
\qquad
\begin{tikzcd} G^{I} \arrow[d] \arrow[r, phantom, "="] & G^{\downarrow} \arrow[d, "(\mathrm{dom}, \mathrm{cod})"] \\ G \times G \arrow[r, phantom, "="] & G \times G \end{tikzcd}$$

Hurewicz

= (iso-) fibration

Path object

= arrow groupoid

Usual H-S Groupoid model .

3) $I = [1]$ in $\underline{sSet = Set^{\Delta^{op}}}$:

Kan fibrations are Hurewicz,
so Id.types follow from the Prop.

4) $I = [1]$ in $\underline{cSet = Set^{\square^{op}}}$:

Hurewicz + Path types
$\Rightarrow$ n-box filling f. all $n > 0$!

4. Box Filling

Def. $f : Y \longrightarrow X$ has <u>n-box filling</u> if for all open boxes $\sqcup \xrightarrow{i^n} I^n$ and all c, b, there's a filler φ, as in:

$$\begin{tikzcd} \sqcup \arrow[r, "b"] \arrow[d, hook, "i^n"'] & Y \arrow[d, "f"] \\ I^n \arrow[r, "c"'] \arrow[ur, dashed, "\exists \varphi"] & X \end{tikzcd}$$

Briefly,

$$i^n \pitchfork f .$$

Prop. Suppose the model $t : \tilde{T} \to T$

(i) has path types,

(ii) is Hurewicz.

Then any type family $A \to X$ classified by t has n-box filling f. all $n > 0$.

Pf.

$$\begin{tikzcd}
\sqcup \arrow[r] \arrow[d, "i_n"'] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\
I^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & X \arrow[r] & T
\end{tikzcd}$$

STS: $i_n \pitchfork t$ $\forall n > 0$.

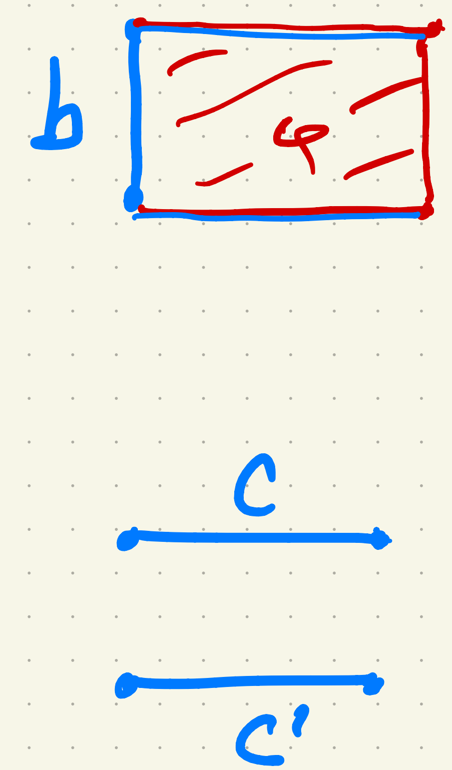
Lemma For any $A \longrightarrow X$,

$$i_n \pitchfork A^I \iff i_{n+1} \pitchfork A .$$

Pf.

$$\begin{tikzcd}
1 \arrow[r, "b"] \arrow[d, "d_0"'] & A^I \arrow[d] \\
I \arrow[r, "c"'] \arrow[ur, dashed, "\varphi"] & A \times A
\end{tikzcd}$$

$i_1 \pitchfork A^I$



$i_2 \pitchfork A$.

Pf. of Prop.

1) t Hurewicz $\Rightarrow$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "i_1"'] & \tilde{T} \arrow[d, "t"] \\ I \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

$i_1 \pitchfork t$,

2) path types $\Rightarrow$

$$\begin{tikzcd} \tilde{T}^I \arrow[r] \arrow[d, "t^I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{T} \arrow[d, "t"] \\ \tilde{T} \times_T \hat{T} \arrow[r] & T \end{tikzcd}$$

$i_1 \pitchfork t^I$,

3) Lemma $\Rightarrow$ $i_2 \pitchfork t$,

Etc., by induction on $n > 0$.

THANKS!

Reference:

S. A. & J. Hua: Path Types in ATT,
arXiv, 2026.