

# Algebraic Type Theory

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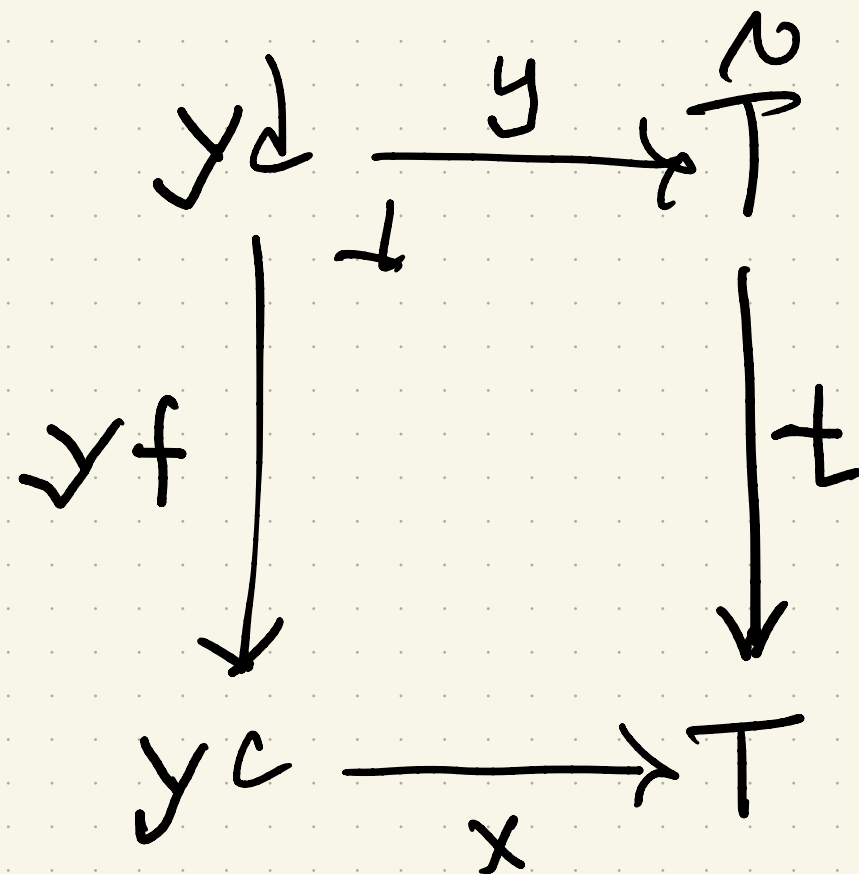
CMU

# 1. Natural Models

Def. A natural model consists of

- a Cat  $\mathcal{C}$
- presheaves  $T, \tilde{T}$
- a natural transformation  
$$t: \tilde{T} \rightarrow T$$
- that's representable

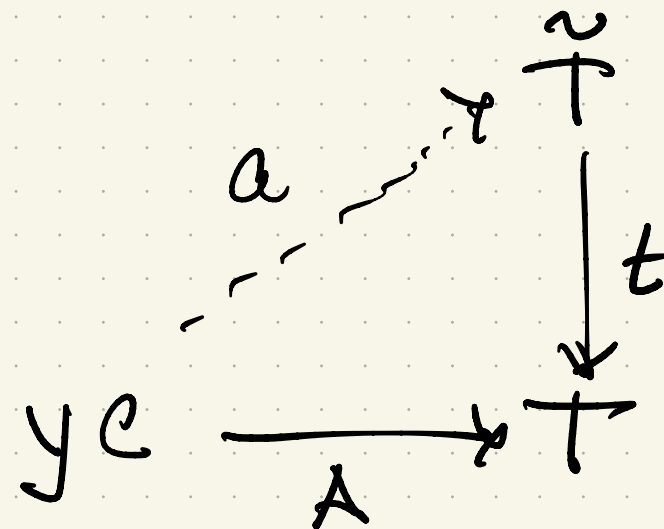
$$\forall c \in \mathcal{C} \quad \forall x \in T_c$$
$$\exists f: d \rightarrow c \quad \exists y \in \tilde{T}_d$$



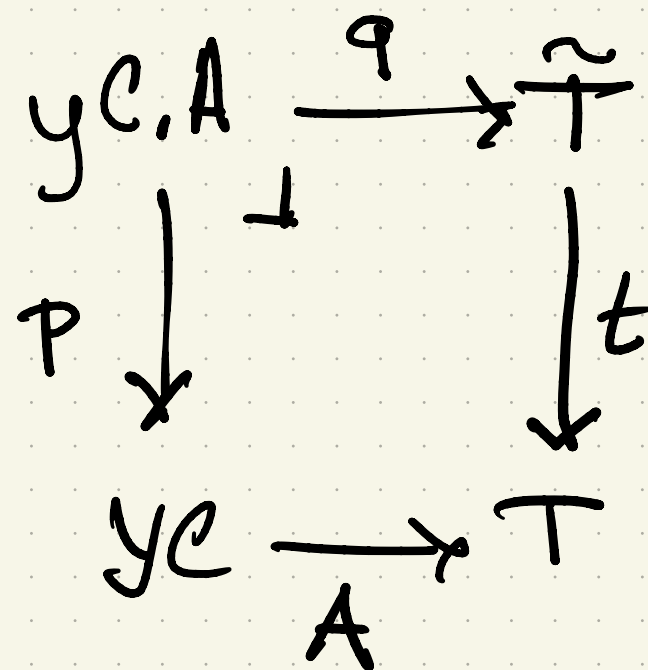
# Remarks

This is equivalent to CwF.

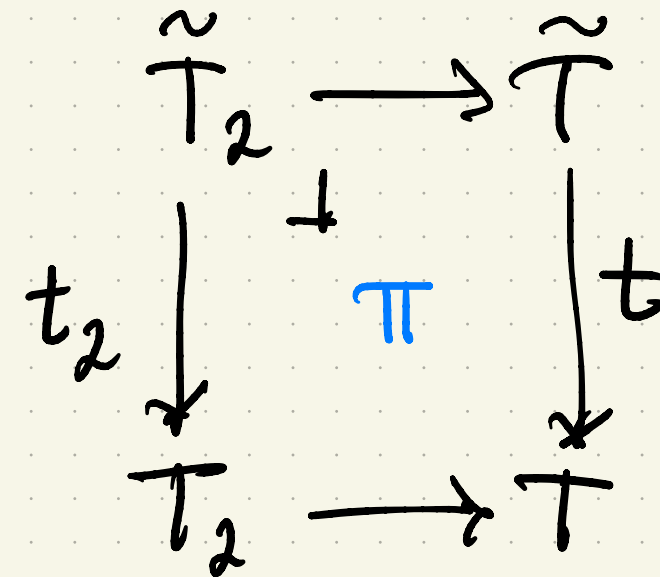
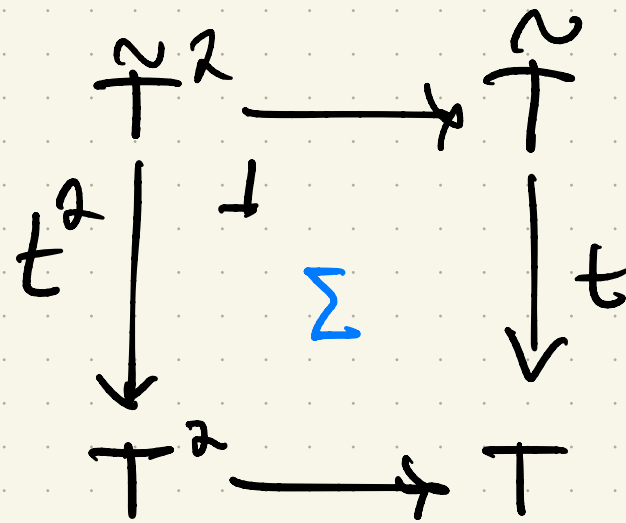
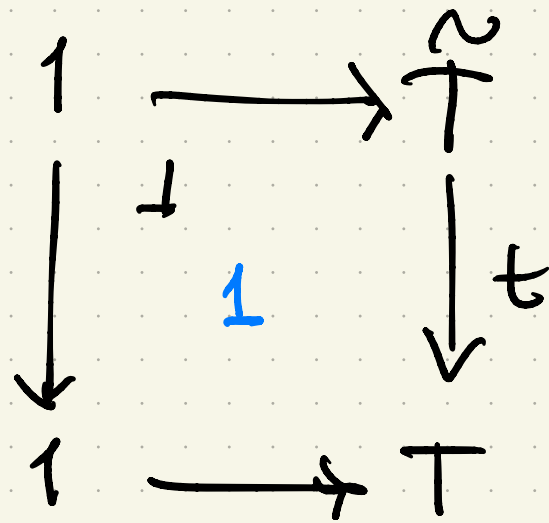
- $\mathcal{C}$  cat of contexts
- $\mathcal{T}$  presheaf of types
- $\tilde{\mathcal{T}}$  presheaf of terms
- Representability of  $t: \tilde{\mathcal{T}} \rightarrow \mathcal{T}$  is context extension



$\mathcal{C} \vdash a: A$



- Type formers  $\perp, \Sigma, \Pi$  are modelled



- We abstract this structure to form that of a  
"Martin-Löf algebra".

## 2. Polynomial Functors

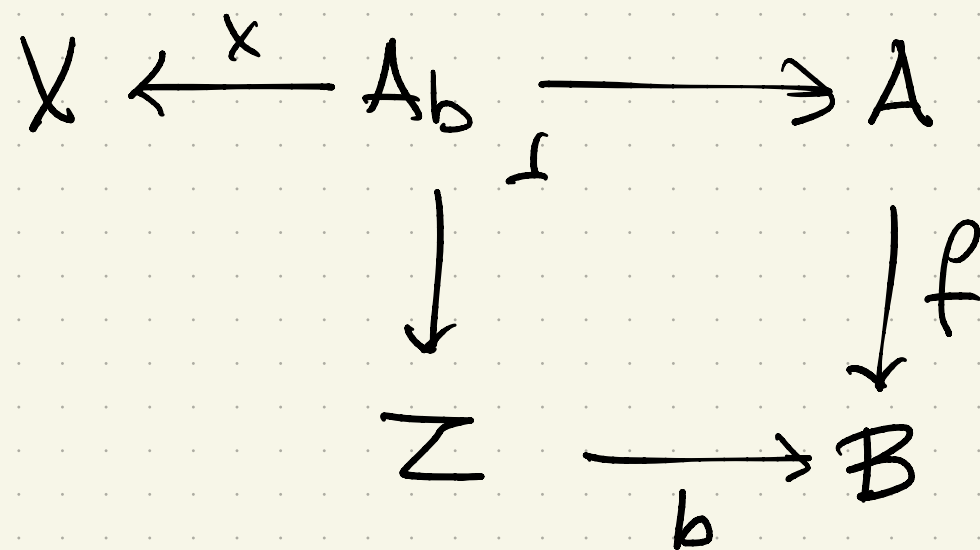
Every  $f: A \rightarrow B$  in an LCCC  $\mathcal{E}$  determines a polynomial functor

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{P_f} & \mathcal{E} \\
 A^* \downarrow & & \uparrow B! \\
 \mathcal{E}/A & \xrightarrow{f_*} & \mathcal{E}/B
 \end{array}$$

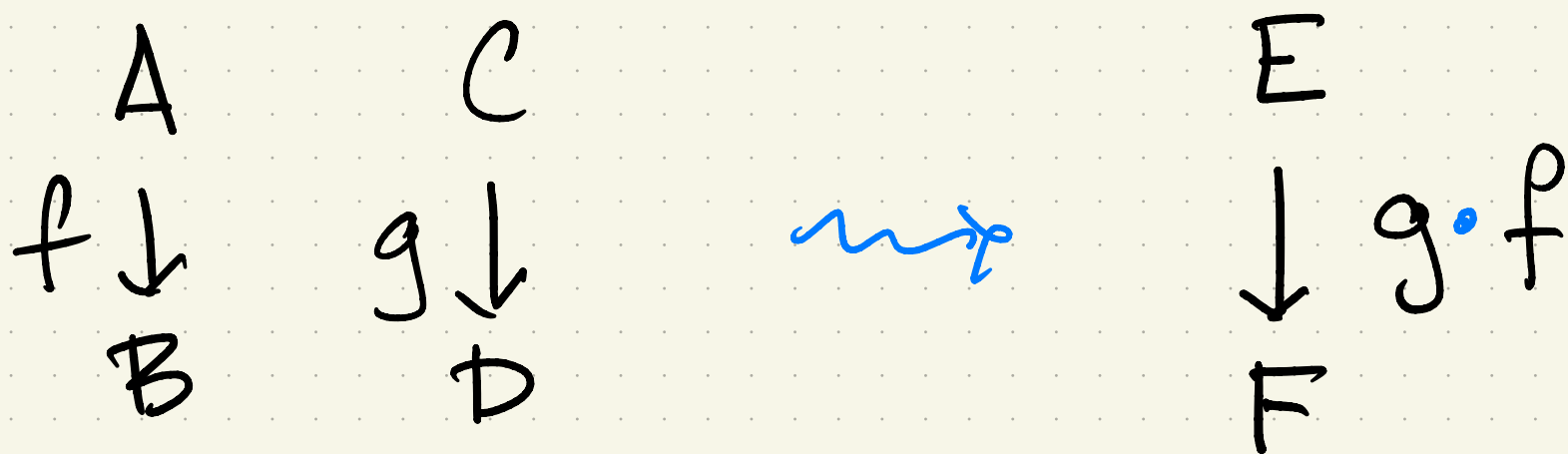
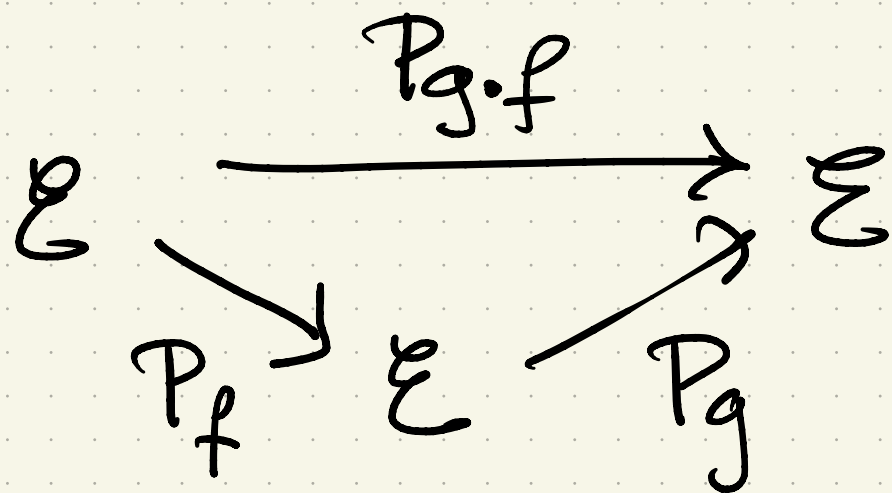
$$\begin{array}{ccc}
 X & \longleftarrow & X \times A \\
 & & \downarrow \\
 & & A \\
 & & \xrightarrow{f} \\
 & & B \\
 & & \downarrow P_f X \\
 & & B
 \end{array}$$

- In the DTT of  $\mathcal{E}$   $P_f X = \sum_{b \in B} X^{A_b}$ .

- The UMP of  $P_f X$  is  $(b, x): Z \longrightarrow P_f X$



- The composite of polynomial functors is polynomial:



- As is  $1_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E}$ , so there is a monoid:

$$(\text{Poly } \mathcal{E}, \cdot, 1_{\mathcal{E}})$$

### 3. M-L Algebras

Def A M-L algebra in a LCC  $\mathcal{E}$  is a map

$$t: \tilde{T} \rightarrow T$$

with structure

$$\begin{array}{ccc} 1 & \xrightarrow{\sim} & \tilde{T} \\ \downarrow & u & \downarrow t \\ 1 & \xrightarrow{\quad} & T \end{array}$$

$$\begin{array}{ccc} \tilde{T}^2 & \xrightarrow{\quad} & \tilde{T} \\ t^2 \downarrow & m & \downarrow t \\ T^2 & \xrightarrow{\quad} & T \end{array}$$

$$\begin{array}{ccc} \tilde{T}_2 & \xrightarrow{\quad} & \tilde{T} \\ t_2 \downarrow & c & \downarrow t \\ T_2 & \xrightarrow{\quad} & T \end{array}$$

where  $P_{t^2} = P_{t \cdot t} = P_t \circ P_t$  and  $t_2 = P_t(t)$ .



- The **unit** determines a natural transformation

$$u: 1_{\mathcal{E}} \rightarrow P_t$$

- The **multiplication** determines another one

$$m: P_t \circ P_t \rightarrow P_t$$

- The **closure** determines an algebra structure

$$c: P_t(t) \rightarrow t$$

## Dominance

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- Basic Example A CWF  $(\mathcal{C}, t: \hat{T} \rightarrow T)$  is a ML-algebra in  $\hat{\mathcal{C}}$  iff it has  $1, \Sigma, \Pi$  as a CWF.

- Thm Let  $t: \hat{T} \rightarrow T$  be a ML-algebra in  $\mathcal{E}$ . Define a CWF  $\hat{t}: T_m \rightarrow T_y$  on  $\mathcal{E}$  by mapping in,

$$\begin{array}{ccc}
 T_m & := & \mathcal{E}(-, \hat{T}) \\
 \hat{t} \downarrow & & \downarrow \\
 T_y & := & \mathcal{E}(-, T) .
 \end{array}$$

Then  $\hat{t}$  has  $1, \Sigma, \Pi$  as a CWF.

Pf: Yoneda preserves ML-algebras.

## 4. Comparison with Clans

Let  $t: \tilde{T} \rightarrow T$  be a natural model in  $\hat{\mathcal{C}}$  and define display maps  $\mathcal{D}_t$  in  $\mathcal{C}$  by:

$$\mathcal{D}_t \ni f \downarrow c \quad \Leftrightarrow \quad \begin{array}{ccc} yd & \xrightarrow{\quad} & \tilde{T} \\ yf \downarrow & \lrcorner & \downarrow t \\ yc & \xrightarrow{\quad} & T \end{array}$$

- Then  $\mathcal{D}_t$  is closed under pullbacks and
  - isos & composition if  $t$  is a **dominance**,
  - pushforwards if  $t$  is **closed**.
- So  $(\mathcal{C}, \mathcal{D}_t)$  is a  **$\Pi$ -clan** if  $t$  is a **ML-algebra**.

Conversely

Thm Given a  $\Pi$ -clan  $(\mathbb{C}, \mathcal{D})$  there's a natural model  $d: \tilde{\mathbb{D}} \rightarrow \mathbb{D}$  in  $\hat{\mathbb{C}}$  that's a ML-algebra, and  $\mathcal{D} = \mathcal{D}_d$ .

Pf

Let

$$\begin{array}{ccc} \tilde{\mathbb{D}} & & \coprod_{y \in \text{dom } f} \\ d \downarrow & ::= & \coprod_{f \in \mathcal{D}} y f \\ \mathbb{D} & & \downarrow \\ & & \coprod_{y \in \text{cod } f} \end{array} \quad .$$

In fact, there's an adjunction\*

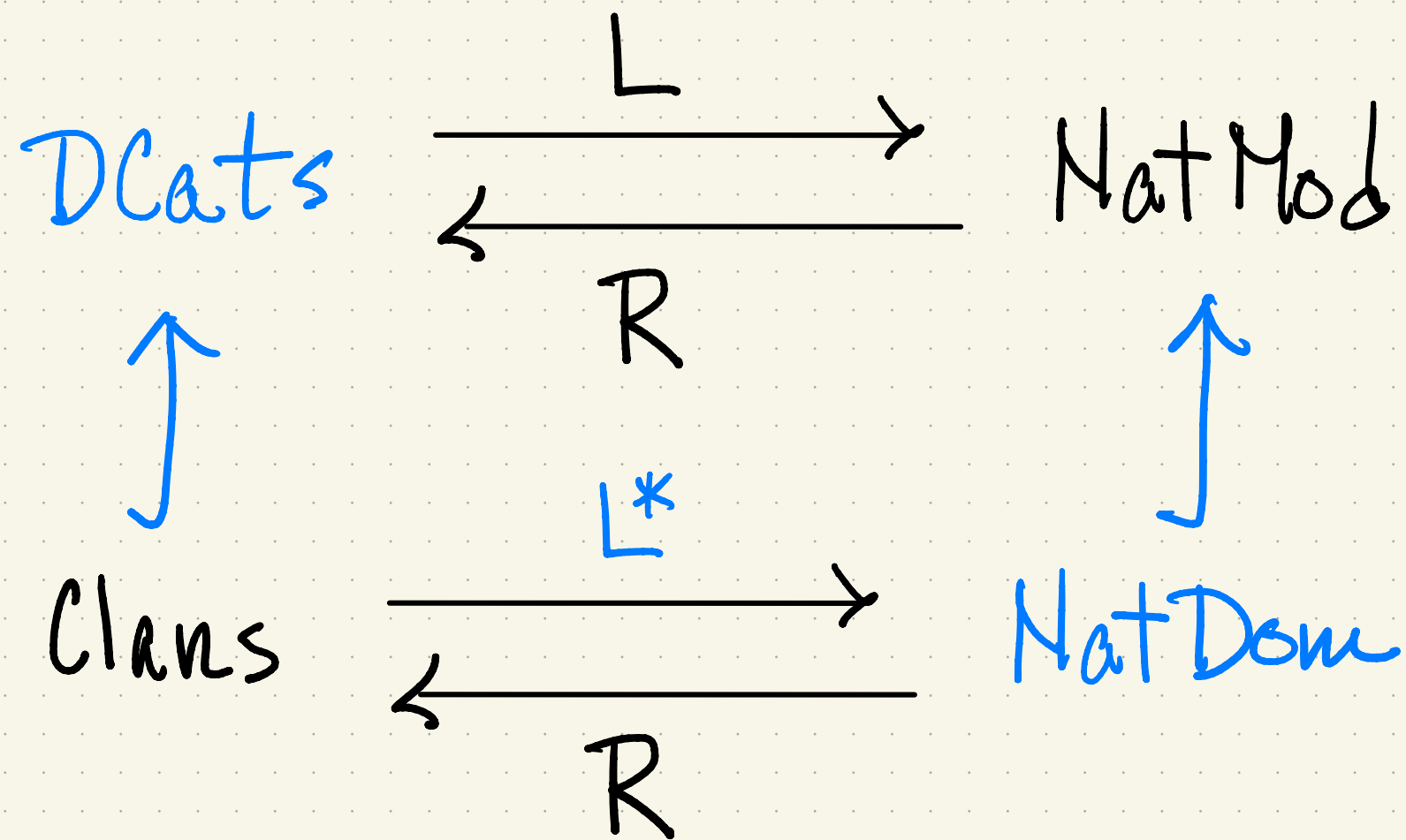
$$\text{Clans} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} \text{Nat Mod}$$

Where

$$L(\mathbb{C}, \mathcal{D}) = \perp\!\!\!\perp y \mathcal{D}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathcal{D}_t) \quad .$$

More accurately



$$L(\mathbb{C}, \mathcal{D}) = \perp\!\!\!\perp y^{\mathcal{D}}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathcal{D}_t)$$

$$L^* = \dots$$

- Given a natural model  $t: \hat{T} \rightarrow T$  we can freely add a "monoid structure"

$$\begin{array}{c}
 t \\
 \downarrow \\
 1 \longrightarrow u \longleftarrow u \cdot u
 \end{array}$$

- This is done by solving the "domain equation"

$$u \cong 1 + t \cdot u$$



- The solution is the colimit of the sequence

$$0 \rightarrow A_t 0 \rightarrow A_t^2 0 \rightarrow \dots$$

for the endofunctor  $A_t x = 1 + t \cdot x$  on  $\text{Poly}(\hat{\mathbb{C}})$ .

- The colimit is  $t^* = 1 + t + t \cdot t + t^{\cdot 3} + \dots$   
 $= \sum_n t^{\cdot n}$ .

- So  $L^*(\mathbb{C}, \mathcal{D}) := L(\mathbb{C}, \mathcal{D})^*$ .

The proof uses 2 lemmas.

Lemma 1 If  $t: \hat{T} \rightarrow T$  is representable

then the polynomial endofunctor

$$P_t: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

has a right adjoint, and so preserves all colimits.

Lemma 2 If  $(\mathbb{C}, \mathcal{D})$  is a clan, then in

$$\begin{array}{ccc} (\mathbb{C}, \mathcal{D}) & \xrightarrow{\eta} & RL(\mathbb{C}, \mathcal{D}) \\ & \searrow^{\eta^*} & \downarrow \\ & & RL^*(\mathbb{C}, \mathcal{D}) \end{array}$$

the unit  $\eta^*$  is an equivalence.

THANKS!



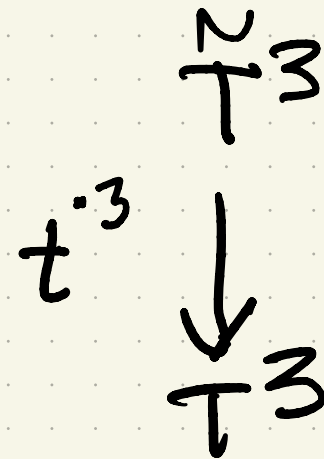
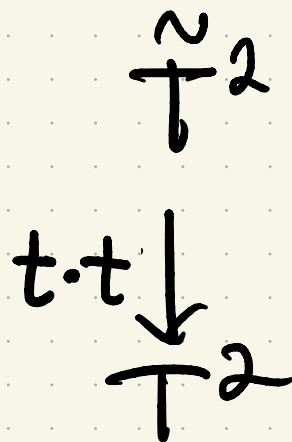
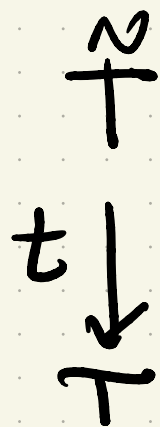
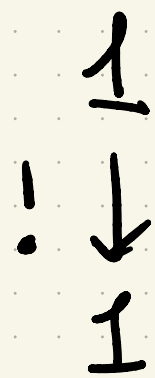
Note  $t^* = \Sigma t^n$  is the free completion of the  
"type theory"  $t: \mathbb{T} \rightarrow T$  under  $\Sigma$ -types.

Consider the maps

$$\begin{array}{ccccccc} & \mathbb{1} & & \mathbb{T} & & \mathbb{T}^2 & & \mathbb{T}^3 & & \dots \\ ! & \downarrow & & t \downarrow & & t \cdot t \downarrow & & t^{\cdot 3} \downarrow & & \dots \\ & \mathbb{1} & & T & & T^2 & & T^3 & & \dots \end{array}$$

as classifying types

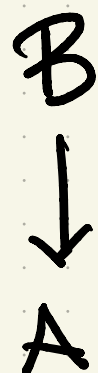
# Maps



...

# classify

A



...

# Contexts

A

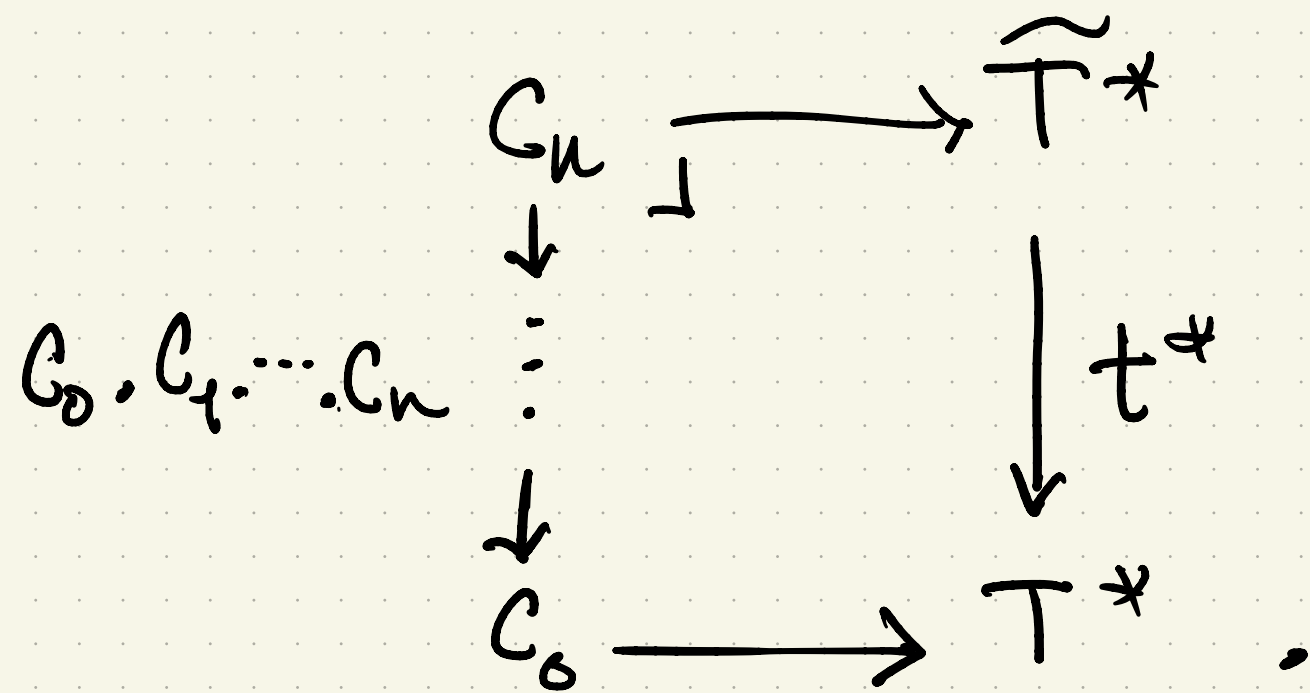
A.B

A.B.C

A.B.C.D

...

Thus  $t^*: \tilde{T}^* \rightarrow T^*$  classifies contexts of  $t$



the theory of contexts  $t^*$  of a theory  $t$  freely adds

$\Sigma$ -types .