Algebraic Type Theory

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1. Natural Models

Def. A <u>natural model</u> consists of

- a cat $\mathbb{C}$
- presheaves $T, \tilde{T}$
- a natural transformation

  $$t : \tilde{T} \to T$$

- that's <u>representable</u>

  $$\forall c \in \mathbb{C} \quad \forall x \in T_c$$

  $$\exists_{f : d \to c} \quad \exists y \in \tilde{T}_d$$

$$\begin{tikzcd}
yd \arrow[r, "y"] \arrow[d, "yf"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\
yc \arrow[r, "x"'] & T
\end{tikzcd}$$

Remarks

This is equivalent to CwF.

- $\mathbb{C}$ cat of contexts
- T presheaf of types
- $\tilde{T}$ presheaf of terms
- Representability of $t : \tilde{T} \to T$ is context extension

$$\begin{tikzcd} & \tilde{T} \arrow[d, "t"] \\ yC \arrow[ur, dashed, "a"] \arrow[r, "A"'] & T \end{tikzcd}$$

$$C \vdash a : A$$

$$\begin{tikzcd} yC.A \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\ yC \arrow[r, "A"'] & T \end{tikzcd}$$

- Type formers $1, \Sigma, \Pi$ are modelled

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "1"] & \tilde{T} \arrow[d, "t"] \\
1 \arrow[r] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{T}^2 \arrow[r] \arrow[d, "t^2"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\Sigma"] & \tilde{T} \arrow[d, "t"] \\
T^2 \arrow[r] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{T}_2 \arrow[r] \arrow[d, "t_2"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\Pi"] & \tilde{T} \arrow[d, "t"] \\
T_2 \arrow[r] & T
\end{tikzcd}
$$

- We abstract this structure to form that of a "Martin-Löf algebra".

2. Polynomial Functors

Every $f : A \longrightarrow B$ in an LCCC $\mathcal{E}$ determines a

polynomial functor

$$\begin{tikzcd}
\mathcal{E} \arrow[rrr, "P_f"] \arrow[dr, "A^*"'] & & & \mathcal{E} \\
& \mathcal{E}/A \arrow[r, "f_*"] & \mathcal{E}/B \arrow[ur, "B_!"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X & X \times A \arrow[l] \arrow[d] & & P_f X \arrow[d] \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

- In the <u>DTT of $\mathcal{E}$</u> $\qquad P_f X = \sum_{b:B} X^{A_b}$.

- The <u>UMP</u> of $P_f X$ is

$$
\begin{array}{c}
(b,x) : Z \longrightarrow P_f X \\
\hline\hline
\begin{tikzcd}
X & A_b \arrow[l, "x"'] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
 & Z \arrow[r, "b"'] & B
\end{tikzcd}
\end{array}
$$

.

- The composite of polynomial functors is polynomial:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "P_{g\cdot f}"] \arrow[dr, "P_f"'] & & \mathcal{E} \\ & \mathcal{E} \arrow[ur, "P_g"'] & \end{tikzcd}$$

$$\begin{matrix} A & C \\ f\downarrow & g\downarrow \\ B & D \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} E \\ \downarrow g\cdot f \\ F \end{matrix}$$

- As is $1_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$, so there is a monoid:

$$(\mathrm{Poly}\,\mathcal{E}, \cdot, 1_{\mathcal{E}})$$

3. M-L Algebras

Def A M-L algebra in a LCCC $\mathcal{E}$ is a map

$$t : \tilde{T} \longrightarrow T$$

with structure

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "u"] & \tilde{T} \arrow[d, "t"] \\
1 \arrow[r] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{T}^2 \arrow[r] \arrow[d, "t^2"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "m"] & \tilde{T} \arrow[d, "t"] \\
T^2 \arrow[r] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{T}_2 \arrow[r] \arrow[d, "t_2"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "c"] & \tilde{T} \arrow[d, "t"] \\
T_2 \arrow[r] & T
\end{tikzcd}$$

where $P_{t^2} = P_{t \circ t} = P_t \circ P_t$ and $t_2 = P_t(t)$.

- the unit determines a natural transformation

$$u : 1_{\mathcal{E}} \longrightarrow P_t$$

- The multiplication determines another one

$$m : P_t \circ P_t \longrightarrow P_t$$

- the closure determines an algebra structure

$$c : P_t(t) \longrightarrow t$$

Dominance

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- Basic Example A CwF $(\mathbb{C}, t : \tilde{T} \to T)$ is a ML-algebra in $\hat{\mathbb{C}}$ iff it has $1, \Sigma, \Pi$ as a CwF.

- Thm Let $t : \tilde{T} \longrightarrow T$ be a ML-algebra in $\mathcal{E}$. Define a CWF $\hat{t} : Tm \to Ty$ on $\mathcal{E}$ by mapping in,

$$\begin{array}{ccc} Tm & := & \mathcal{E}(-, \tilde{T}) \\ {\scriptstyle \hat{t}}\downarrow & & \downarrow \\ Ty & := & \mathcal{E}(-, T) \end{array}.$$

Then $\hat{t}$ has $1, \Sigma, \Pi$ as a CwF.

Pf: Yoneda preserves ML-algebras.

4. Comparison with Clans

Let $t : \tilde{T} \to T$ be a natural model in $\hat{\mathbb{C}}$ and define <u>display maps</u> $\mathcal{D}_t$ in $\mathbb{C}$ by:

$$\mathcal{D}_t \ni f \left\downarrow \begin{matrix} d \\ \\ c \end{matrix}\right. \iff \begin{tikzcd} yd \arrow[r] \arrow[d, "yf"'] \arrow[dr, phantom, "\lrcorner" very near start] & \tilde{T} \arrow[d, "t"] \\ yc \arrow[r] & T \end{tikzcd}.$$

- Then $\mathcal{D}_t$ is closed under pullbacks and
    - isos & composition if t is a dominance,
    - pushforwards if t is closed.
- So $(\mathbb{C}, \mathcal{D}_t)$ is a Π-clan if t is a ML-algebra.

Conversely

Thm Given a Π-clan $(\mathbb{C}, \mathcal{D})$ there's a natural model $d : \tilde{D} \longrightarrow D$ in $\widehat{\mathbb{C}}$ that's a ML-algebra, and $\mathcal{D} = \mathcal{D}_d$.

Pf Let

$$\begin{tikzcd} \tilde{D} \arrow[d, "d"'] \\ D \end{tikzcd} := \coprod_{f \in \mathcal{D}} yf \qquad \begin{tikzcd} \coprod y\,\mathrm{dom} f \arrow[d] \\ \coprod y\,\mathrm{cod} f \end{tikzcd} .$$

In fact, there's an adjunction*

$$\mathrm{Clans} \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{NatMod}$$

where

$$L(\mathbb{C}, \mathfrak{D}) = \coprod y\mathfrak{D}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathfrak{D}_t) \quad .$$

More accurately

$$
\begin{tikzcd}
\mathbf{DCats} \arrow[r, shift left, "L"] & \mathbf{NatMod} \arrow[l, shift left, "R"] \\
\mathbf{Clans} \arrow[u, hook] \arrow[r, shift left, "L^*"] & \mathbf{NatDom} \arrow[u, hook] \arrow[l, shift left, "R"]
\end{tikzcd}
$$

$$L(\mathbb{C}, \mathfrak{D}) = \coprod y\mathfrak{D}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathfrak{D}_t)$$

$$L^* = \ldots$$

- Given a natural model $t : \tilde{T} \to T$ we can freely add a "monoid structure"

$$\begin{tikzcd} & t \arrow[d] & \\ 1 \arrow[r] & u & u \cdot u \arrow[l] \end{tikzcd}$$

- This is done by solving the "domain equation"

$$u \cong 1 + t \cdot u .$$

- The solution is the colimit of the sequence

$$0 \to A_t 0 \to A_t^2 0 \to \cdots$$

for the endofunctor $A_t X = 1 + t \cdot X$ on $\mathrm{Poly}(\hat{\mathbb{C}})$.

- The colimit is $t^* = 1 + t + t \cdot t + t^{\cdot 3} + \cdots$

$$= \sum_n t^{\cdot n} .$$

- So $\mathbb{L}^*(\mathbb{C}, \mathbb{D}) := \mathbb{L}(\mathbb{C}, \mathbb{D})^*$.

The proof uses 2 lemmas.

Lemma 1 If $t : \tilde{T} \longrightarrow T$ is representable then the polynomial endofunctor

$$P_t : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

has a right adjoint, and so preserves all colimits.

Lemma 2 If $(\mathbb{C}, \mathcal{D})$ is a clan, then in

$$\begin{tikzcd}
(\mathbb{C}, \mathcal{D}) \arrow[r, "\eta"] \arrow[dr, "\sim", "\eta^*"'] & RL(\mathbb{C}, \mathcal{D}) \arrow[d] \\
& RL^*(\mathbb{C}, \mathcal{D})
\end{tikzcd}$$

the unit η^* is an equivalence.

THANKS!

Note $t^* = \Sigma t^{\cdot n}$ is the free completion of the "type theory" $t : \tilde{T} \longrightarrow T$ under Σ-types.

Consider the maps

$$\begin{array}{ccccccccc} \underline{1} & & \tilde{T} & & \tilde{T}^2 & & \tilde{T}^3 & & \\ {\scriptstyle !}\downarrow & & {\scriptstyle t}\downarrow & & {\scriptstyle t \cdot t}\downarrow & & {\scriptstyle t^{\cdot 3}}\downarrow & & \cdots \\ 1 & & T & & T^2 & & T^3 & & \end{array}$$

as classifying types

Maps

$$
\begin{tikzcd}
1 \arrow[d, "!"'] & \tilde{T} \arrow[d, "t"'] & \tilde{T}^2 \arrow[d, "t \cdot t"'] & \tilde{T}^3 \arrow[d, "t^{\cdot 3}"'] & \cdots \\
1 & T & T^2 & T^3 &
\end{tikzcd}
$$

classify

$$
\begin{tikzcd}
 & & & D \arrow[d] & \\
 & & C \arrow[d] & C \arrow[d] & \\
 & B \arrow[d] & B \arrow[d] & B \arrow[d] & \\
A & A & A & A & \cdots
\end{tikzcd}
$$

Contexts

$A \qquad A.B \qquad A.B.C \qquad A.B.C.D \qquad \cdots$

Thus $t^* : \widetilde{T}^* \to T^*$ classifies <u>contexts of t</u>

$$\begin{tikzcd}
 & C_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{T}^* \arrow[dd, "t^*"] \\
C_0.C_1.\cdots.C_n & \vdots \arrow[d] & \\
 & C_0 \arrow[r] & T^*
\end{tikzcd}$$

.

<u>the theory of contexts</u> t^* of a theory t freely adds

Σ-types .