The isotropy group of a first-order theory

Steve Awodey
jww
Spencer Breiner
Pieter Hofstra

Category Theory
Octoberfest
2022

The Isotropy Group of a Topos

(cf. Funk-Hofstra-Steinberg 2012)

Let $\mathcal{T}$ be a topos & consider the functor

$$Z : \mathcal{T}^{op} \longrightarrow \mathrm{Grp}$$

$$Z(X) = \mathrm{Aut}\,(X^* : \mathcal{T} \longrightarrow \mathcal{T}/X).$$

So $Z(X)$ consists of natural automorphisms of the pullback functor:

$$\mathcal{T} \ni F \longmapsto \begin{array}{c} F \times X \\ \downarrow \pi_2 \\ X \end{array} \in \mathcal{T}/X$$

A map $X \xleftarrow{f} Y$ acts by whiskering with pullback:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "X^*"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{T}/X \arrow[r, "f^*"] & \mathcal{T}/Y \end{tikzcd}$$

$$Z(X) \ni \alpha \longrightarrow f^* \cdot \alpha \in Z(Y)$$

Briefly:

$$\alpha \in Z(X) = \mathrm{Aut}(X^*)$$

$$X^* : \mathcal{T} \xrightarrow{\circlearrowleft\alpha} \mathcal{T}/X \,.$$

Prop. The functor $\mathcal{Z} : \mathcal{X}^{op} \longrightarrow \mathrm{Grp}$ is representable,

$$\mathcal{Z} \cong \mathcal{X}(-, Z) \, ,$$

for a group object Z in $\mathcal{X}$, called the isotropy group of $\mathcal{X}$.

Remark The group $Z_{\mathcal{X}}$ also acts on the $\mathcal{E}$-valued points $p : \mathcal{E} \longrightarrow \mathcal{X}$, for any topos $\mathcal{E}$, in the following sense: For any global $\alpha : 1 \longrightarrow Z$ we have a natural automorphism:

$$1^* : \mathcal{X} \xrightarrow{\ \alpha\ } \mathcal{X}/1 \simeq \mathcal{X} \, .$$

So for each $F \in \mathcal{X}$ we have an iso

$$\begin{tikzcd} F \arrow[r, "\alpha_F", "\sim"'] \arrow[d] & F \arrow[d] \\ E \arrow[r, "\sim", "\alpha_E"'] & E \end{tikzcd}$$

natural in F !

And moreover, for all $X \in \mathcal{I}$,

$$\begin{tikzcd} \mathcal{I} \arrow[r, "\alpha_1 \circlearrowleft 1_{\mathcal{I}}"] \arrow[dr, "X^*"', "\alpha_X \circlearrowleft"] & \mathcal{I} \arrow[d, "X^*"] \\ & \mathcal{I}/X \end{tikzcd}$$

and

$$\alpha_X = X^* \cdot \alpha_1$$

since α is natural in X.

Conversely, given any natural automorphism

$$1_{\mathcal{I}} : \mathcal{I} \xrightarrow{\alpha \circlearrowleft} \mathcal{I} .$$

Whiskering by <u>any</u> $p : \mathcal{E} \longrightarrow \mathcal{I}$ results in

$$\mathcal{E} \xrightarrow{\alpha \cdot p \circlearrowleft} \mathcal{I} .$$

<u>Prop.</u> There's a group isomorphism:

$$\Gamma Z_{\mathcal{I}} \xrightarrow{\sim} \mathrm{Aut}(1_{\mathcal{I}})$$

between global sections of $Z_{\mathcal{I}}$ and the <u>center of $\mathcal{I}$</u> : the group of natural automorphisms of the identity $1_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$.

Now suppose that $\mathcal{X} = \mathrm{Set}[\mathbb{T}]$ classifies $\mathbb{T}$-models. Then for any topos $\mathcal{E}$ we get an action of (sections of) the group $Z_{\mathbb{T}}$ of $\mathrm{Set}[\mathbb{T}]$ on $\mathbb{T}$-models in $\mathcal{E}$, since

$$\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}) \simeq \mathrm{Top}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]),$$

and we just saw that $\pi^* Z_{\mathbb{T}}$ acts naturally on the RHS.

This leads to the following idea:

Prop. (A-B 2012) Let $S[\mathbb{T}]$ be a classifying topos for a theory $\mathbb{T}$, with universal $\mathbb{T}$-model $\mathcal{U}_{\mathbb{T}}$. Then the isotropy group $Z_{S[\mathbb{T}]}$ agrees with the internal automorphism group of the $\mathbb{T}$-model $\mathcal{U}_{\mathbb{T}}$,

$$Z_{S[\mathbb{T}]} \cong \mathrm{Aut}(\mathcal{U}_{\mathbb{T}}).$$

Internal vs. External

- The external group of sections

$$\Pi Z_{\mathbb{T}} \text{ in } \mathbf{Set}$$

acts naturally on all $\mathbb{T}$-models (in all toposes):

$$\alpha \in \Pi Z_{\mathbb{T}} \qquad \begin{tikzcd} M \arrow[r, "\alpha_M", "\sim"'] \arrow[d, "h"'] & M \arrow[d, "h"] \\ N \arrow[r, "\sim", "\alpha_N"'] & N \end{tikzcd}$$

- The internal group object

$$Z_{\mathbb{T}} \text{ in } \mathcal{S}[\mathbb{T}]$$

does something more:

For any model M (in any $\mathcal{E}$), there's a stalk $(Z_{\mathbb{T}})_M$ s.th.

- for $\alpha \in \Pi_{\mathcal{E}}(Z_{\mathbb{T}})_M$ there's
- and for any $h : M \to N$ there's $\alpha_h : N \to N$ s.th.

$$\begin{tikzcd} M \arrow[r, "\alpha_M", "\sim"'] \arrow[d, "h"'] & M \arrow[d, "h"] \\ N \arrow[r, "\sim", "\alpha_h"'] & N \end{tikzcd}.$$

Logical Schemes

Let $\mathbb{T}$ be a coherent theory, and $\mathcal{E}_{\mathbb{T}}$ the classifying pretopos, so the classifying topos is:

$$\mathcal{S}[\mathbb{T}] = \mathrm{Sh}(\mathcal{E}_{\mathbb{T}}) .$$

In $\mathcal{S}[\mathbb{T}]$ there is a pretopos $\tilde{\mathcal{E}}_{\mathbb{T}}$, given by strictifying the stack:

$$\tilde{\mathcal{E}}_{\mathbb{T}} : \mathcal{E}_{\mathbb{T}}^{op} \longrightarrow \mathrm{Cat}$$

$$\tilde{\mathcal{E}}_{\mathbb{T}}(X) = \mathcal{E}_{\mathbb{T}}/X ,$$

corresponding to the codomain fibration

$$\begin{array}{c} \mathcal{E}_{\mathbb{T}}^{\to} \\ \mathrm{cod}\downarrow \\ \mathcal{E}_{\mathbb{T}} \end{array} .$$

Prop. $\tilde{\mathcal{E}}_{\mathbb{T}}$ is a sheaf representation of $\mathcal{E}_{\mathbb{T}}$, in the sense $\Gamma\tilde{\mathcal{E}}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}}$.

Groupoid of Models

We then make $\tilde{\mathcal{E}}_{\mathbb{T}}$ into an equivariant sheaf on the groupoid of $\mathbb{T}$-models :

$$\mathbb{G}_{\mathbb{T}} = G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}} \quad .$$

Where :

$X_{\mathbb{T}}$ = Space of $\mathbb{T}$-models

$G_{\mathbb{T}}$ = Space of $\mathbb{T}$-model isos .

The groupoid $\mathbb{G}_{\mathbb{T}}$ supports the groupoid representation of $S[\mathbb{T}]$:

$$S[\mathbb{T}] \simeq Sh(\mathcal{E}_{\mathbb{T}})$$

$$\simeq Sh_{eq}(\mathbb{G}_{\mathbb{T}}) \quad ,$$

where $Sh_{eq}(\mathbb{G}_{\mathbb{T}})$ is the topos of $\mathbb{G}_{\mathbb{T}}$ equivariant sheaves on $X_{\mathbb{T}}$.

See: Joyal-Tierney, Butz-Moerdijk, A.-Forssell, Breiner .

Now move the sheaf $\tilde{\mathcal{E}}_{\mathbb{T}}$ across the equivalence

$$\mathrm{Sh}(\mathcal{E}_{\mathbb{T}}) \simeq \mathrm{Sh}_{eq}(\mathcal{G}_{\mathbb{T}})$$

to get an equivariant sheaf on $\mathcal{G}_{\mathbb{T}}$, called the structure sheaf of the logical scheme

$$(\mathcal{G}_{\mathbb{T}}, \tilde{\mathcal{E}}_{\mathbb{T}})$$

of the theory $\mathbb{T}$. (Breiner 2012)

Remark. There's also the constant equivariant sheaf $\Delta\mathcal{E}_{\mathbb{T}}$ on $\mathcal{G}_{\mathbb{T}}$, and a canonical map

$$\varepsilon : \Delta\mathcal{E}_{\mathbb{T}} \longrightarrow \tilde{\mathcal{E}}_{\mathbb{T}},$$

namely the transpose of the equivalence

$$\mathcal{E}_{\mathbb{T}} \xrightarrow{\sim} \Gamma\tilde{\mathcal{E}}_{\mathbb{T}}.$$

<u>Prop.</u> The isotropy group $Z_{\mathbb{T}}$ of $\mathbb{T}$ is isomorphic to the group of automorphisms of ε:

$$Z_{\mathbb{T}} \cong \mathrm{Aut}(\Delta \mathcal{E}_{\mathbb{T}} \to \tilde{\mathcal{E}}_{\mathbb{T}}) .$$

<u>Corollary.</u> The <u>stalk</u> of $Z_{\mathbb{T}}$ at a model M is the group of <u>inner automorphisms</u>: "definable" automorphisms w/parameters from M

$$(Z_{\mathbb{T}})_M \cong \mathrm{Aut}_i(M) .$$

Pf:

$$(Z_{\mathbb{T}})_M \cong \mathrm{Aut}(\Delta \mathcal{E}_{\mathbb{T}} \to \tilde{\mathcal{E}}_{\mathbb{T}})_M$$

$$\cong \mathrm{Aut}(M^* \Delta \mathcal{E}_{\mathbb{T}} \to M^* \tilde{\mathcal{E}}_{\mathbb{T}})$$

$$\cong \mathrm{Aut}(\mathcal{E}_{\mathbb{T}} \to \mathcal{E}_{\mathbb{T}[M]}) \ \ldots$$

$$\begin{tikzcd}
Sh_{eq}(\mathcal{G}_{\mathbb{T}}) \arrow[drr, "\text{stalk@}M"] \arrow[d, phantom, "\wr"] & & \\
Sh(\mathcal{E}_{\mathbb{T}}) \arrow[rr, "M^*"] & & Set \\
\mathcal{E}_{\mathbb{T}} \arrow[u, hook] \arrow[urr, "M"] \arrow[rr] & & \mathcal{E}_{\mathbb{T}}[M] \arrow[u, "\pi"']
\end{tikzcd}$$

where $\mathcal{E}_{\mathbb{T}}[M]$ in the factorization

$$\begin{tikzcd}
\mathcal{E}_{\mathbb{T}} \arrow[rr, "M"] \arrow[dr] & & Set \\
& \mathcal{E}_{\mathbb{T}}[M] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

is:

$$\mathcal{E}_{\mathbb{T}}[M] = (\tilde{\mathcal{E}}_{\mathbb{T}})_M \simeq M^*(\tilde{\mathcal{E}}_{\mathbb{T}})$$

$$= \varinjlim_{SM} \mathcal{E}_{\mathbb{T}}/(-) = \mathcal{E}_{\mathbb{T}[M]} .$$

Thus $\mathbb{T}[M]$ is the theory of M.

The stalk of the isotropy group at M is then:

$$
\begin{aligned}
(Z_{\mathbb{T}})_M &\cong \mathrm{Aut}(\mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}[M]}) \\
&\cong \mathrm{Aut}(\mathcal{U}_{\mathbb{T}[M]}),
\end{aligned}
$$

which is indeed $\mathrm{Aut}_i(M)$, as claimed, since the model $\mathcal{U}_{\mathbb{T}[M]}$ is in the syntactic pretopos $\mathcal{E}_{\mathbb{T}[M]}$, so its automorphisms are in the syntax of $\mathbb{T}[M]$.

Remark. The stalk pretopos $\mathcal{E}_{\mathbb{T}[M]}$ is local, in the sense that its $\Gamma : \mathcal{E}_{\mathbb{T}[M]} \longrightarrow \mathbf{Set}$ is a pretopos functor: 1 is projective & indecomposible.

References

- Funk, Hofstra, Steinberg: Isotropy & Crossed Toposes, TAC 2012.

- Breiner, Spencer: Scheme representation for first-order logic, PhD thesis, CMU 2012.