

Some studies on Postnikov decomposition in
synthetic homotopy theory

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Abstract

In this thesis we will present three studies on synthetic homotopy theory. The first study will deal with higher group theory, and there we will recover and slightly generalize a theorem due in different forms to Mac Lane-Whitehead and Sính, which states that the type of 2-groups is equivalent to a certain type of triples, involving in particular a cohomological invariant of the 2-groups called the Sính invariant. The second study will be on the topic of modalities, where we will show that, assuming any one of a family of variants of the axiom of choice, we can construct a hypercompletion modality (these terms are defined below and in the relevant chapter). The final study is a summary of some contributions we made jointly with Barton, Mörtberg and Ljungström [6] to the formalization of a proof of the Serre finiteness theorem. In the conclusion we will very briefly sketch out a relationship between one of the key objects in the second and the third studies, to strengthen the impression that each of the studies are fairly tightly interwoven, as I suggest at the opening of the thesis.

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Contents

1	Introduction	5
1.1	Preface	5
1.2	Notation	7
1.3	Preliminary material	8
1.3.1	Modality	13
1.3.2	Connected and truncated maps	18
1.3.3	Eilenberg-Mac Lane spaces	20
1.3.4	Homotopy pushouts	23
2	2-groups	26
2.1	Higher group theory	27
2.2	Wärn torsors	29
2.3	Classifying 2-groups	30
2.4	Further discussion	35
2.5	Classifying some untruncated ∞ -groups	37
3	Hypercompletion	46

3.1	Introduction	46
3.2	Preliminaries	47
3.3	Countable limits	50
3.4	Choice principles	58
3.5	Postnikov effectiveness and hypercompletion	59
3.6	Dependent choice in synthetic Stone duality	64
3.7	Conclusion	66
4	Serre finiteness	67
4.1	Connected covers	67
4.1.1	Identifying the Fibers of the Whitehead Tower	70
4.2	Stably almost finite spaces	72
4.2.1	Stably almost finite spaces and related notions	74
4.2.2	Applications to the Serre finiteness theorem	79
5	Conclusion	85
5.1	Connections	86
5.2	Loose ends	87
6	Bibliography	90

Chapter 1

Introduction

1.1 Preface

The content of this thesis amounts to three essentially independent research projects. All three have resulted in publications, two of which are co-authored, I try to give credit to my collaborators where that is appropriate below, they are also thanked in the acknowledgements.

Despite their essential independence, there is one animating theme running through the three studies. This is the concept of the Postnikov decomposition of a space, which is defined below. I only realized very late in the day, that the three projects, which I had thought were all completely independent, were really united by this one concept, and so it is not always foregrounded as explicitly as it should be in the text of this thesis. Nevertheless, the reader looking for a unifying idea should bear it in mind as they read.

We will spend the remainder of the introduction introducing some notation, definitions and preliminary lemmas, drawn from the literature, which will be used in the studies proper. In the introduction, we have been a bit haphazard about including proofs. This haphazardness is somewhat offset by regular references, either to the relevant literature, or to Agda code formalizing the statements and their proofs that we are discussing. Still the referencing in this introduction is at times a little sloppy, we generally give a relevant reference at the beginning of a section, but not always at the location of a particular statement, so we should emphasize: none of the material in chapter one is original.

In general, throughout the thesis we include hyperlinks to Agda code where we think the code is appropriate or illuminating. These links are always marked with a  symbol (intended to suggest certification?) at the statement of the theorem or the beginning of a proof, if the proof does not appear immediately after a relevant statement. In the second study the code linked is entirely our own, and the stamp is in black. In the third study the code should all be considered joint work with Reid Barton, Anders Mörtberg and Axel Ljungström, the stamp there is blue. In the first study the code was written by Perry Hart, based on the authors notes and paper proofs, the hyperlinks are only included for the readers interest, and that code should not be considered an integral part of this thesis, these stamps are in red.

1.2 Notation

We work in homotopy type theory [4, 34, 42, 44]. We make ready use of the elimination rule (J -rule) for intensional identity types, the univalence axiom, and higher inductive types.

We mostly follow the notation of (e.g.) [43]. So if P is a type-family over A , then the product type is $(a : A) \rightarrow P(a)$, and the sum type is $(a : A) \times P(a)$. We will write $\mathcal{U}$ for the universe of all (small) types, $\mathcal{U}^{\leq -1}$ for the type of propositions and $\mathcal{U}^{\leq n}$ for the type of n -types. We might write $\mathcal{U}_{>n}$ for the universe of n -connected types, and $\mathcal{U}_{\text{pt}}$ for the universe of pointed types. We allow ourselves to combine these subscripts and superscripts at times, so that, for example, $\mathcal{U}_{>n, \text{pt}}^{\leq m}$ is the universe of m -truncated, n -connected, pointed types.

If X is a pointed type, pointed by x , we write $\Omega(X, x)$ for the loop space of X , or just ΩX when there is no confusion about the point. We will write $\|X\|_n$ for the n -truncation of X , and sometimes $\|X\|$ for the propositional truncation (omitting a subscript -1), we will write $|\bullet|_n$ for the universal map $X \rightarrow \|X\|_n$ for any type X , and sometimes $|\bullet|$ for the universal map into the propositional truncation. If we want to indicate that a function or an equivalence is pointed (i.e. is between two pointed types and preserves the point) then we will write $X \rightarrow_{\text{pt}} Y$ or $X \simeq_{\text{pt}} Y$. We will usually write $X \simeq Y$ for the type of equivalences, but sometimes we write $\text{Aut}(X)$ instead of $X \simeq X$ when we want to think of the latter as a (higher) group (see below).

The unit type will always be denoted 1 , and $\mathbb{N}$ will always denote the naturals.

If $f : X \rightarrow Y$, and $y : Y$, then $\text{fib}_f(y)$ denotes the fiber of f at y , the type $(x : X) \times f(x) = y$.

Along with the n -truncations, we will use a small stock of higher inductive types, mostly in the final chapter of the thesis. In particular, we make use of higher inductive types constructed from pushouts, such as the suspension $\text{Susp}X$ and the join $X * Y$. We write $\pi_k X$ for the k th homotopy group of X , the 0-truncation of the k th iterated loop space of X .

1.3 Preliminary material

Theorem 1 (Fundamental theorem of identity types).  If A is pointed by a and $P : A \rightarrow \mathcal{U}$ is a family of types such that $P(a)$ is pointed, and $(x : A) \times P(x)$ is contractible with the point of $P(a)$ as the center of contraction, then $P(x) \simeq (x = a)$ naturally in x .

Proof. Theorem 11.2.2 in [\[34\]](#). □

Remark 1. It follows from the detail of the proof that if we have a pointed function $f : (X, x) \rightarrow_{\text{pt}} (Y, y)$, and we have $g : P(x) \rightarrow \Omega(Y, y)$ such that

$$g(p) = \text{proj}_2(f)^{-1} \cdot \text{ap}_{\text{proj}_1(f)}(e^{-1}(p)) \cdot \text{proj}_2(f)$$

then, if e is the equivalence established by the theorem, $g \circ e = \Omega(f)$.

Proposition 1 (Whitehead's Lemma).  If X and Y are n -truncated spaces, then $f : X \rightarrow Y$ is an equivalence if and only if $\|f\|_0 : \|X\|_0 \rightarrow \|Y\|_0$ is an

equivalence and $\pi_n(f) : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ is an equivalence of groups for all $n > 0$.

Proof. This appears as Theorem 8.8.3 in [42]. □

Definition 1 (Pullback). *If $f : A \rightarrow B \leftarrow C : g$ is a cospan of maps, then the pullback $A \times_B C$ is the type $(a : A) \times (c : C) \times (f(a) = g(c))$.*

Proposition 2 (Universal property for pullbacks). *Fix a co-span $f : A \rightarrow B \leftarrow C : g$, for any type X , we have the following equivalence*

$$(X \rightarrow A \times_B C) \simeq (h : X \rightarrow A) \times (k : X \rightarrow B) \times f \circ h = g \circ k$$

Proof. This follows from the type-theoretic axiom of choice [42], along with function extensionality. □

Proposition 3 (Pasting for pullbacks). *Suppose we have the following commutative diagram:*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

and suppose that the right-most square is a pullback diagram, then the left-most square is a pullback diagram if and only if the outer rectangle is a pullback diagram.

Proof. This follows from the universal property for pullbacks. □

Remark 2. If $f : X \rightarrow Y$ and $p : x_1 =_X x_2$, $q : x_3 =_X x_4$, $r : f(x_1) = f(x_3)$, then

$$\text{transport}_{(p,q)}^{\lambda(x,x').f(x)=f(x')}(r) = \text{ap}_f(p)^{-1} \cdot r \cdot \text{ap}_f(q)$$

The remark can be proved by an easy path induction.

Proposition 4 (3×3 for homotopy pullbacks). Consider a commuting (up-to-homotopy) diagram like so:

$$\begin{array}{ccccc} R & \xrightarrow{f_1} & S & \xleftarrow{f_2} & T \\ \downarrow g_1 & & \downarrow g_3 & & \downarrow g_5 \\ U & \xrightarrow{f_3} & V & \xleftarrow{f_4} & W \\ \uparrow g_2 & & \uparrow g_4 & & \uparrow g_6 \\ X & \xrightarrow{f_5} & Y & \xleftarrow{f_6} & Z \end{array}$$

The iterated pullback of the co-span of verticals is the same as the iterated pullback of the co-span of horizontals.

Proof. This proof is modelled on the proof of Theorem 3.3.15 in [30].

We give names to the homotopies witnessing the commutativity of the diagram:

$$\begin{aligned} h_1 : g_3 \circ f_1 &= f_3 \circ g_1 & h_2 : g_3 \circ f_2 &= f_4 \circ g_5 \\ h_3 : g_4 \circ f_5 &= f_3 \circ g_2 & h_4 : g_4 \circ f_6 &= f_4 \circ g_6 \end{aligned}$$

We also give names to the pullbacks of the horizontal and vertical co-spans.

The horizontal pullbacks:

$$P_1 := (r : R) \times (t : T) \times (f_1(r) = f_2(t))$$

$$P_2 := (u : U) \times (w : W) \times (f_3(u) = f_4(w))$$

$$P_3 := (x : X) \times (z : Z) \times (f_5(x) = f_6(z))$$

The vertical pullbacks:

$$Q_1 := (r : R) \times (x : X) \times (g_1(r) = g_2(x))$$

$$Q_2 := (s : S) \times (y : Y) \times (g_3(s) = g_4(y))$$

$$Q_3 := (t : T) \times (z : Z) \times (g_5(t) = g_6(z))$$

Now we have functions

$$G_1 : P_1 \rightarrow P_2$$

$$G_1 = \lambda(r, t, p).(g_1(r), g_5(t), h_1(r)^{-1} \cdot \text{ap}_{g_3}(p) \cdot h_2(t))$$

$$G_2 : P_3 \rightarrow P_2$$

$$G_2 = \lambda(x, z, p).(g_2(x), g_6(z), h_3(x)^{-1} \cdot \text{ap}_{g_4}(p) \cdot h_4(z))$$

$$F_1 : Q_1 \rightarrow Q_2$$

$$F_1 = \lambda(r, x, p).(f_1(r), f_5(x), h_1(r) \cdot \text{ap}_{f_3}(p) \cdot h_3(x)^{-1})$$

$$F_2 : Q_3 \rightarrow Q_2$$

$$F_2 = \lambda(t, z, p).(f_2(t), f_6(z), h_2(t) \cdot \text{ap}_{f_4}(p) \cdot h_4(z)^{-1})$$

Now observe:

$$\begin{aligned}
& ((r, t, p) : P_1) \times ((x, z, q) : P_3) \times (G_1(r, t, p) = G_2(x, z, p)) \\
& \simeq ((r, t, p) : P_1) \times ((x, z, q) : P_3) \times ((i, j) : g_1(r) = g_2(x) \times g_5(t) = g_6(z)) \\
& \quad \times \left[h_1(r)^{-1} \cdot \text{ap}_{g_3}(p) \cdot h_2(t) \stackrel{\lambda(u, v) \cdot f_3(u) = f_4(v)}{=}_{(i, j)} h_3(x)^{-1} \cdot \text{ap}_{g_4}(q) \cdot h_4(z) \right] \\
& \simeq ((r, t, p) : P_1) \times ((x, z, q) : P_3) \times ((i, j) : g_1(r) = g_2(x) \times g_5(t) = g_6(z)) \\
& \quad \times [\text{ap}_{f_3}(i)^{-1} \cdot h_1(r)^{-1} \cdot \text{ap}_{g_3}(p) \cdot h_2(t) \cdot \text{ap}_{f_4}(j) = h_3(x)^{-1} \cdot \text{ap}_{g_4}(q) \cdot h_4(z)] \\
& \simeq ((r, x, i) : Q_1) \times ((t, z, j) : Q_3) \times ((p, q) : f_1(r) = f_2(t) \times f_5(x) = f_6(z)) \\
& \quad \times [\text{ap}_{g_3}(p)^{-1} \cdot h_1(r) \cdot \text{ap}_{f_3}(i) \cdot h_3(x)^{-1} \cdot \text{ap}_{g_4}(q) = h_2(t) \cdot \text{ap}_{f_4}(j) \cdot h_4(z)^{-1}] \\
& \simeq ((r, x, i) : Q_1) \times ((t, z, j) : Q_3) \times ((p, q) : f_1(r) = f_2(t) \times f_5(x) = f_6(z)) \\
& \quad \times \left[h_1(r) \cdot \text{ap}_{f_3}(i) \cdot h_3(x)^{-1} \stackrel{\lambda(s, y) \cdot g_3(s) = g_4(y)}{=}_{(p, q)} h_2(t) \cdot \text{ap}_{f_4}(j) \cdot h_4(z)^{-1} \right] \\
& \simeq ((r, x, i) : Q_1) \times ((t, z, j) : Q_3) \times (F_1(r, x, i) = F_2(t, z, j))
\end{aligned}$$

The result now follows by univalence. $\square$

Proposition 5 (Theorem 8.4.6 in [42]). *Let $A \rightarrow B \rightarrow C$ be a fiber sequence, then there is a long exact sequence of homotopy groups:*

$$\dots \longrightarrow \pi_{n+1}C \longrightarrow \pi_n A \longrightarrow \pi_n B \longrightarrow \pi_n C \longrightarrow \pi_{n-1}A \longrightarrow \dots$$

Proof. We apply the pasting lemma in the special case where the right-most square is a fiber sequence pullback diagram. $\square$

1.3.1 Modality

Modalities will be the central object of study in the third chapter of this thesis, so we outline the basic theory in modalities in HoTT now.

The following definitions, propositions and examples come from Rijke, Shulman, and Spitters [36].

Definition 2 (Modality). *Suppose $\bigcirc : \mathcal{U} \rightarrow \mathcal{U}$ is an operation on the universe with a unit $\eta : (X : \mathcal{U}) \rightarrow X \rightarrow \bigcirc X$, we say that the pair $(\bigcirc, \eta)$ is a (uniquely eliminating) modality if the map*

$$\lambda f.f \circ \eta_X : ((x : \bigcirc X) \rightarrow \bigcirc P(x)) \rightarrow ((x : X) \rightarrow \bigcirc P(\eta_X(x)))$$

is an equivalence for all $X : \mathcal{U}$ and $P : \bigcirc X \rightarrow \mathcal{U}$.

Definition 3 (Nullification). *If $(\bigcirc, \eta)$ is a modality, and X is a type, we say that X is nullified by the modality if $\bigcirc X$ is contractible.*

Definition 4. *If $(\bigcirc, \eta)$ is a modality and $f : A \rightarrow B$ is a map, we say that f is nullified by the modality if all its fibers are nullified in the sense of the previous definition.*

Definition 5 (Σ -closed reflective subuniverse). *A Σ -closed reflective subuniverse is a triple $(M, \bigcirc, \eta)$ where $\bigcirc$ and η are an operation and a unit as before, and M is a function taking types to propositions, such that*

1. $M(\bigcirc X)$ holds for all types X

2. For any type X and any type B such that $M(B)$, the map

$$\lambda f.f \circ \eta_X : (\bigcirc X \rightarrow B) \rightarrow (X \rightarrow B)$$

is an equivalence

3. If X is any type, and $P : X \rightarrow \mathcal{U}$, and $M(X)$, and $M(P(x))$ for all $x : X$, then

$$M((x : X) \times P(x))$$

Proposition 6 (Functoriality of $\bigcirc$). *If $(M, \bigcirc, \eta)$ is a reflective subuniverse, then there is an operation $(A, B : \mathcal{U}) \rightarrow (A \rightarrow B) \rightarrow (\bigcirc A \rightarrow \bigcirc B)$ which for convenience we shall label $\bigcirc$ also. This is functorial in the sense that*

- $\bigcirc(\text{id}_A) = \text{id}_{\bigcirc A}$, and
- $\bigcirc(g \circ f) = \bigcirc g \circ \bigcirc f$

Proof. Lemma 1.21 in [36]. □

Proposition 7. *If $(M, \bigcirc, \eta)$ and $(M', \bigcirc', \eta')$ are two Σ -closed reflective subuniverses, then they are equal in the type of reflective closed subuniverses if and only if (the proposition) $(M(X) \Leftrightarrow M'(X))$ is inhabited for all types X .*

Proof. Theorem 1.18 in [36]. □

Definition 6. A (pullback-)stable orthogonal factorization system is a pair of predicates

$$L, R : (A, B : \mathcal{U}) \rightarrow (A \rightarrow B) \rightarrow \mathcal{U}^{\leq -1}$$

such that

- For any $f : A \rightarrow B$, and $g : B \rightarrow C$, if $L(f)$ and $L(g)$, then $L(g \circ f)$, and likewise for R .
- If $f : A \rightarrow B$ is an isomorphism, then $L(f)$ and $R(f)$.
- For all $f : A \rightarrow B$, the type

$$(C : \mathcal{U}) \times (g : A \rightarrow C) \times (h : C \rightarrow B) \times (f = hg) \times L(g) \times R(h)$$

of factorizations of the map into an L -map and an R -map is contractible, and such factorization diagrams are preserved by pullbacks.

Proposition 8. If (L, R) is a stable orthogonal factorization system, then R is precisely the type of maps which satisfy right-lifting with respect to all maps in L and vice versa.

Proof. Lemma 1.46 in [36]. □

Theorem 2 (Modalities and reflective subuniverses). The type of modalities is equivalent to the types of

1. Σ -closed reflective subuniverses

2. Stable orthogonal factorization systems

where, in particular

1. M is the proposition that $\eta(X)$ is an equivalence, if this holds we say that X is a modal type.
2. The maps L are those which are nullified by the modality. The maps in R are those with modal fibers; the units η always lie in L .

Hence, in particular, a modality is uniquely determined by the subuniverse of types such that η is an equivalence.

Proof. Using Theorems 1.13, 1.16, 1.34 and 1.54 in [36]. □

Example 1 (n -Truncation). Fix a natural number $n : \mathbb{N}$, the pair

$$(\lambda X. \|X\|_n, \lambda X. |\bullet|_n)$$

is a modality.

The corresponding subuniverse is the type of n -truncated types, and the left maps in the corresponding factorization system are the n -connected maps.

Definition 7 (Left-exact modality). If $(\circlearrowleft, \eta)$ is a modality, and $B \rightarrow A \leftarrow C$ is a cospan of types, then there is a canonical map

$$(\circlearrowleft B \times_{\circlearrowleft A} \circlearrowleft C) \rightarrow \circlearrowleft (B \times_A C)$$

We say that $(\bigcirc, \eta)$ is a left-exact modality if this map is an equivalence for all spans.

One consequence is that the maps which are nullified by the modality coincide with the maps whose image under $\bigcirc$ is an equivalence.

Proposition 9. *The 0-truncation modality is not left-exact. This proposition generalizes to all n .*

Proof. Consider the pullback square:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & S^1 \end{array}$$

$\|\mathbb{Z}\|_0 = \mathbb{Z}$, but $\|S^1\|_0 = 1$, so we would be asking for an equivalence between 1 and $\mathbb{Z}$.

More generally we can take a pullback square:

$$\begin{array}{ccc} K(\mathbb{Z}, n) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & K(\mathbb{Z}, n+1) \end{array}$$

And similar reasoning applies. □

We'll return to modalities in the second study, for now we'll present some more useful facts about truncated and connected maps.

1.3.2 Connected and truncated maps

Proposition 10. *If*

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ \downarrow f & & \downarrow g \\ B & \xrightarrow{v} & Y \end{array}$$

commutes, and the right-most map is n -truncated, and the left-most map is n -connected, then the type of fillers for the square is contractible. In fact, this characterizes n -connected maps.

Proof. Refer to Proposition [8](#), Theorem [2](#) and Example [1](#). □

Proposition 11. *If X and Y are n -truncated spaces, then maps $f : X \rightarrow Y$ are n -truncated.*

Proof. It suffices to prove that if X is an n -type and $P : X \rightarrow \mathcal{U}$ is a family of n -types (in our case we have X and the family $\lambda x. f(x) = y$), then $(x : X) \times P(x)$ is an n -type. But recall that if (x, p) and (x', p') are terms of the latter type, then

$$((x, p) = (x', p')) = (q : x = x') \times (\text{transport}_q^P(p) = p')$$

so we can conclude by induction (since the claim is obvious for $n = -2$). (Indeed, this is also part of the content of Example [1](#)). □

Proposition 12. *If A is n -connected and $P(a)$ is n -connected for each $a : A$, then $(a : A) \times P(a)$ is also n -connected.*

Proof. Follows immediately from Theorem 7.3.9 in [\[42\]](#). □

Proposition 13. *If $f : A \rightarrow B$ is n -connected, then $\|f\|_n$ is an equivalence. The converse is not necessarily true (c.f. Definition [7](#) and Proposition [9](#)). However, $f : \|A\|_n \rightarrow \|B\|_n$ is an equivalence if and only if f is n -connected.*

Proof. For an example of a map such that $\|f\|_0$ is an equivalence, but the map is not 0-connected, see the proof of Proposition [9](#). The map $1 \rightarrow S^1$ is as specified. $\square$

Proposition 14. *Suppose $f = g \circ h$ and f and h are n -connected, then so is g .*

Proof. Immediate using the orthogonal factorization system. $\square$

Proposition 15. *If A is $(n+1)$ -connected, then $x =_A y$ is n -connected for all $x, y : A$. If f is $(n+1)$ -connected, then all $\text{ap}(f)$ are n -connected.*

As a partial converse: if A is merely inhabited and $x =_A y$ are all n -connected, then A is $(n+1)$ -connected. If f is surjective (i.e. (-1) -connected) and all $\text{ap}(f)$ are n -connected, then f is $(n+1)$ -connected.

Proof. The first statement follows from Theorem 7.3.12 in [\[42\]](#).

According to (amongst others) Leclerc [\[21\]](#) p. 8], the converse is due to Coquand. If A satisfies the hypothesis of the converse, then $\|A\|_{n+1}$ is a proposition, using the theorem cited above. Then $\|A\|_{n+1} \simeq \|A\|_{-1}$ and the latter type is contractible by assumption. The corresponding statement for maps follows readily. $\square$

Proposition 16. *Suppose $f = g \circ h$ and f and g are $(n+1)$ -connected, then h is n -connected.*

Proof. Follows from Proposition 15. □

Corollary 1. *If $\|f\|_{n+1}$ is an equivalence, then f is n -connected.*

Proposition 17. *For any X , the map $|\bullet|_n : X \rightarrow \|X\|_n$ is n -connected.*

Proof. This follows from Example 1. □

1.3.3 Eilenberg-Mac Lane spaces

Eilenberg-Mac Lane spaces were introduced to HoTT in Finster and Licata 18, they will crop up in the first and final chapters of this thesis, and so we spend a little time on them now.

Definition 8 (Classifying Space). *If G is a group, then the classifying space BG is a higher inductive type with a point constructor $\text{pt} : BG$, path constructors $\text{loop}(g) : \text{pt} =_{BG} \text{pt}$ for each $g : G$, and higher path constructors:*

$$\text{ident} : \text{loop}(e) = \text{refl}_{\text{pt}}$$

$$\text{compt}(g, h) : \text{loop}(gh) = \text{loop}(g) \cdot \text{loop}(h) \text{ for each } g, h : G$$

$$\text{trnct}(a, b, p, q, r, s) : r = s \text{ for each } a, b : BG, p, q : a = b \text{ and } r, s : p = q.$$

Proposition 18. $G \simeq \Omega(BG, \text{pt})$

Proof. This is a consequence of Proposition 1 and appears as Theorem 2.3 in 18.

Finster and Licata show that the underlying function for the equivalence is loop. □

The next result follows from the induction principles for classifying spaces presented in [18].

Proposition 19. *The classifying space BG for a group G satisfies the universal property that for any pointed 1-type (X, x) the type of pointed maps $f : (BG, \text{pt}) \rightarrow (X, x)$ is equivalent to the type of group homomorphisms $\tilde{f} : G \rightarrow \pi_1(X, x)$.*

Definition 9. *If G is a group, then the n th Eilenberg-Mac Lane space for G is defined inductively:*

$$\begin{aligned} K(G, 1) &:= BG \\ K(G, n + 1) &:= \|\text{Susp}^n(BG)\|_{n+1} \end{aligned}$$

Combining Proposition [1] with Proposition [19] and the induction rule for suspensions presented in [42], we arrive at the following corollary. We have formalized a proof, which we present here for completeness.

Corollary 2 (to Proposition [1]).  *If X is pointed, n -connected and $n + 1$ -truncated, then it is equal to the Eilenberg-Mac Lane space $K(\pi_n(X), n)$.*

Proof. To apply Whitehead's lemma, we aim to find a pointed map

$$f : K(\pi_n(X), n) \rightarrow X$$

such that $\pi_n(f)$ is an isomorphism, the rest is immediate. Notice we have the following chain of natural equivalences of function types:

$$\begin{aligned} K(\pi_n(X), n) &\rightarrow X \simeq \text{Susp}^{n-1}(B\pi_n(X)) \rightarrow X \\ &\simeq B\pi_n(X) \rightarrow \Omega^{n-1}X \\ &\simeq \pi_n(X) \rightarrow \pi_n(X) \end{aligned}$$

Given $f : K(\pi_n(X), n) \rightarrow X$, the image of f under the first equivalence is $f \circ |\bullet|_n$, the image under the second equivalence is $\Omega^{n-1}(f \circ |\bullet|_n) \circ \eta^{n-1}$ where η is the unit of the loop-suspension adjunction. Finally, the image under the third equivalence is:

$$\Omega(\Omega^{n-1}(f \circ |\bullet|_n) \circ \eta^{n-1}) = \Omega^n(f \circ |\bullet|_n) \circ \Omega(\eta^{n-1}) = \pi_n(f) \circ \Omega(\eta^{n-1})$$

The map $\Omega(\eta^{n-1})$ is an isomorphism by the Freudenthal suspension theorem (8.6.4 in [42]): Since the types $\text{Susp}^k(B\pi_n(X))$, with $0 < k \leq n-1$ are always (at least) 1-connected, so is the composite of the η s and applying Ω then gives us a 0-connected, 0-truncated map – i.e. an equivalence.

So, it follows that, if the image of f under the chain of equivalences is an isomorphism, then so is $\pi_n(f)$ by 3-for-2, and indeed we can use the natural equivalences, applied successively in the other direction to choose f so that its image is the identity, which finishes the proof. $\square$

1.3.4 Homotopy pushouts

The other higher inductive types we will make use of in this thesis are constructed from pushouts, and we'll describe some of their features here, many of these results, and much more, appear in [11, 42].

Definition 10. *If $A \xleftarrow{f} B \xrightarrow{g} C$ is a span of maps, the pushout is a higher inductive type with the following constructors*

$$\text{inl} : A \rightarrow A +_B C$$

$$\text{inr} : C \rightarrow A +_B C$$

$$\text{push} : (b : B) \rightarrow \text{inl}(f(b)) = \text{inr}(g(b))$$

Proposition 20 (Universal property for pushouts). *If $A \xleftarrow{f} B \xrightarrow{g} C$ is a span of maps, and X is any type, then there is an equivalence of types*

$$(A +_B C \rightarrow X) \simeq (h : A \rightarrow X) \times (k : C \rightarrow X) \times ((x : X) \rightarrow h \circ \text{inl} = k \circ \text{inr})$$

Proof. This is part of the definition of the higher inductive type. □

Definition 11 (Pasting for pushouts). *Suppose we have the following commutative diagram:*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

and suppose that the left-most square is a pushout diagram, then the right-

most square is a pushout diagram if and only if the outer rectangle is a pushout diagram.

Proof. Using the universal property. □

Proposition 21 (3×3 for homotopy pushouts). *Suppose we have the following commutative diagram*

$$\begin{array}{ccccc}
 A & \longleftarrow & B & \longrightarrow & C \\
 \uparrow & & \uparrow & & \uparrow \\
 D & \longleftarrow & E & \longrightarrow & F \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \longleftarrow & H & \longrightarrow & K
 \end{array}$$

The iterated pushout of the columns is equivalent to the iterated pushouts of the rows.

It's possible to give a similar proof to that of Proposition 4, but we only refer to this statement once below, and in an inessential way, so we omit the proof here.

Definition 12. *If $f : A \rightarrow B$ the cofiber of f is defined to be the pushout of the span $1 \leftarrow A \rightarrow B$.*

Proposition 22.  *Suppose we have a commutative square*

$$\begin{array}{ccc}
 A & \longrightarrow & B \\
 \downarrow & & \downarrow \\
 C & \longrightarrow & D
 \end{array}$$

and the vertical maps are $(n-1)$ and n -connected respectively, then the induced maps between the cofibers of the horizontal maps is n -connected.

Definition 13. Suppose X and Y are two types, the join of X and Y , $X * Y$ is defined to be the pushout of the span $X \leftarrow X \times Y \rightarrow Y$.

Definition 14. If $f : W \rightarrow Y$ and $g : X \rightarrow Z$, then $f * g : W * X \rightarrow Y * Z$ is the obvious map coming from the universal property of the pushout. This operation is functorial.

Proposition 23 (Barton and Campion [5]).  If $f : W \rightarrow Y$ is n_1 -connected, $g : X \rightarrow Z$ is n_2 -connected, W is m_1 -connected and Z is m_2 -connected, then $f * g$ is at least $(\min[n_1 + m_2, n_2 + m_1] + 2)$ -connected.

Proposition 24.  If $X \rightarrow Y \rightarrow Z$ is a cofiber sequence, then so is $Y \rightarrow Z \rightarrow \text{Susp}X$.

Chapter 2

2-groups

In the previous chapter, we saw something of the relationship between groups and homotopy types, through the prism of the Eilenberg-Mac Lane spaces. It turns out that this principle, that information about a (set-level) group can be captured through a certain higher type has very general applications [\[7\]](#), and moreover allows the concept of groups to be generalized to spaces of, in principle, any homotopy dimension.

After two sections summarizing some general theory, this chapter describes our small contributions to this broad and significant field in synthetic mathematics: synthetic higher group theory. Some of our results were formalized by Perry Hart as we wrote our paper together [\[20\]](#). The scope of that paper is somewhat different from this chapter, but there is a lot of overlap between the final sections of that paper and the first section of this chapter. I have added hyperlinks at the relevant locations. The stamp is in red to distinguish it from those formal

theorems which the author I myself formalized or partly formalized, to make it clear that Hart wrote all the code at the relevant locations.

2.1 Higher group theory

Definition 15. *Let X be a pointed type, a delooping of X is a pointed type Y such that $\Omega Y = X$ as pointed types.*

Definition 16 (Buchholtz, van Doorn, and Rijke [10]). *A higher group is a pointed type equipped with a connected delooping.*

Proposition 25. *If X is a pointed set, then connected deloopings of X correspond to group structures on X such that the point of X is the identity element.*

Proof. Suppose Y is a connected delooping of X , then Y is a pointed, connected 1-type, hence $Y = B\pi_1 Y$, so $\pi_1 Y = X$ with some group structure. The unit element must be the point of X , because the equivalence $\Omega Y \simeq X$ is pointed.

Conversely, suppose $(X, \cdot, e)$ is a group structure on X and e is the point of X . Then $Y := B(X, \cdot, e)$ is a pointed connected type and $\Omega Y = X$, as required.

These operations are inverses using earlier results. $\square$

Proposition 26. *If X is a pointed set, and Y is a connected delooping of X , then Y must be 1-truncated.*

Proof. Since Y is connected, this follows immediately from the equation $\Omega Y = X$. $\square$

Proposition 27. *If X is a pointed type with a 1-truncated delooping, then X must be a set.*

Proof. This follows immediately, since identity types on 1-types are always sets.

□

We generalize

Definition 17 (Buchholtz, van Doorn, and Rijke [10]). *An n -group is a pointed type with a connected, n -truncated delooping.*

Proposition 28 (Buchholtz, van Doorn, and Rijke [10]). *The type of n -groups is equivalent to the type of pointed, n -truncated, connected types.*

Proof. We are asking about the type of n -groups defined like so:

$$(X : \mathcal{U}_{\text{pt}}) \times (Y : \mathcal{U}_{>0, \text{pt}}^{\leq n}) \times (X = \Omega Y)$$

and we can re-associate and cancel to obtain the result.

□

In particular, a 2-group is a type with a pointed, 2-truncated, connected delooping. The type of 2-groups is equivalent to the type of pointed, 2-truncated, connected types.

Other characterizations of the category of 2-groups are given in the classical literature, we focus on one, due in different forms to [26] and [40]. They classify 2-groups in terms of triples consisting of:

- A group G

- An abelian group H with a G -action
- A cohomology class $c \in H^3(G; H)$.

We will recover a form of this result in HoTT, and discuss extensions to n -groups for larger n .

2.2 W rn torsors

In this section we summarize relevant definitions and theorems of [45]. We assume throughout that X is a pointed type.

Definition 18. *The type of X -torsors is the type:*

$$TX := (Y : \mathcal{U}) \times \|Y\| \times ((y : Y) \rightarrow X \simeq_{\text{pt}} (Y, y))$$

Definition 19. *The type of pointed X -torsors is the type:*

$$((Y, y, \mu) : TX) \times Y$$

Theorem 3 (W rn [45]).  *If the type of pointed X -torsors is contractible, then*

1. *The type of torsors has a point $(Y, |y|, \mu)$*
2. *The type of torsors is the unique connected delooping of X , that is to say:*

$$\Omega(TX, (Y, |y|, \mu)) = X$$

and if $\Omega Z = X$ and Z is connected, then there is an equivalence

$$(TX, (Y, |y|, \mu)) \simeq_{\text{pt}} Z$$

3. For any X -torsor (Y', y', μ')

$$((Y', y', \mu') =_{TX} (Y, |y|, \mu)) = Y'$$

Proof sketch. The proof uses the fundamental theorem of identity types. $\square$

Corollary 3.  For any abelian group A and any $n \geq 0$, there is an equivalence

$$K(A, n+1) \simeq_{\text{pt}} T(K(A, n))$$

2.3 Classifying 2-groups

The type of 2-groups is equivalently the type $\mathcal{U}_{>0, \text{pt}}^{\leq 2}$ of pointed, connected 2-types 10.

Proposition 29.  There is an equivalence of types:

$$\mathcal{U}_{>0, \text{pt}}^{\leq 2} \simeq ((B : \mathcal{U}_{>0, \text{pt}}^{\leq 1}) \times (F : B \rightarrow \mathcal{U}_{>1}^{\leq 2}) \times F(\text{pt}_B))$$

Proof. Suppose A is a pointed connected 2-type. We can find a triple (B, F, p) with the type B a pointed connected 1-type; the map $F : B \rightarrow \mathcal{U}_{>1}^{\leq 2}$; and the

point $p : F(\text{pt}_B)$. For B we take $\|A\|_1$ the 1-truncation of A . We know we have the truncation map $|\bullet|_1 : A \rightarrow B$. For F we take the map $\lambda b. \text{fib}_{|\bullet|_1}(b)$. The point of B is $|\text{pt}_A|_1$ and we can take $p := (\text{pt}_A, \text{refl})$. (To see that F has the desired properties – observe $|\bullet|_1$ is 1-connected, and $F(b) = (a : A) \times |a|_1 = b$, which is a 2-type because A is a 2-type and each $|a|_1 = b$ is a set). Call this map from pointed connected 2-types to triples $\mathbb{F}$.

Given (B, F, p) we can find a pointed, connected 2-type. Namely

$$(b : B) \times (Fb)$$

with the point (pt_B, p) . Call this map from triples to pointed connected 2-types $\mathbb{G}$.

To see that these maps are mutually inverse, observe:

$$\|(b : B) \times (Fb)\|_1 \simeq \|(b : B) \times \|(Fb)\|_1\|_1 \simeq \|B\|_1 \simeq B$$

This is because (Fb) is 1-connected and B is a 1-type. Call this equivalence e . I claim $e(|x|_1) = \text{fst}(x)$ for all $x : (b : B) \times (Fb)$. Observe $x = (b, f)$ for some $(b : B)$ and $(f : Fb)$.

$$e(|b, f|_1) = e(|(b, f)|_1) = e'(|(b, |f|_1)|_1) = e''(|b|_1) = b = \text{fst}(b, f)$$

Where e' and e'' are the equivalences $(\|(b : B) \times \|(Fb)\|_1\|_1 \simeq \|B\|_1 \simeq B)$ and $(\|B\|_1 \simeq B)$ respectively. These equivalences are all pointed, as are the maps

$|\bullet|_1$ and fst . Hence, the fiber of $|\bullet|_1$ over any point y is the fiber of fst over $e(y)$, which is $F(e(y))$, and the point is still the point. So $\mathbb{F}\mathbb{G}(B, F, p) = (B, F, p)$, using univalence, i.e. $\mathbb{F} \circ \mathbb{G} = \text{id}$.

Meanwhile, there is a pointed equivalence $e : A \simeq (b : ||A||_1) \times (\text{fib}_{|\bullet|_1}(b))$ given by $\lambda a. (|a|_1, (a, \text{refl}))$. It is an equivalence because if $(b, (x, q))$ is a point in $(b : ||A||_1) \times (\text{fib}_{|\bullet|_1}(b))$, then the fiber of e over $(b, (x, q))$ is contractible, as

$$\begin{aligned} \text{fib}_e(b, (x, q)) &\simeq (a : A) \times [(|a|_1, (a, \text{refl})) = (b, (x, q))] \\ &\simeq (((a, p) : (a : A) \times (a = x)) \times (r : |a|_1 = b)) \\ &\quad \times [r = \text{ap}_{|\bullet|_1}(p) \cdot q] \\ &\simeq (r : |x|_1 = b) \times [r = q] \end{aligned}$$

which is contractible.

This uses some easy facts about identity types of sigma types, and also the fundamental theorem of identity types. Actually this is just an instance of the “straightening-unstraightening” principle.

Anyway, this shows that $\mathbb{G} \circ \mathbb{F} = \text{id}$.

This proves the equivalence □

Proposition 30 (Buchholtz, van Doorn and Rijke [\[10\]](#)). *There is an equivalence of types:*

$$\mathcal{U}_{>0, \text{pt}}^{\leq 1} \simeq \text{Grp}$$

In one direction the equivalence sends a group G to its delooping BG .

Proof. Uses material above. □

Proposition 31.  Given a pointed type X , an abelian group A and $n \geq 1$, there is an equivalence of types:

$$(X \simeq_{\text{pt}} K(A, n+1)) \simeq (n\text{Conn}(X) \times (n+1)\text{Trunc}(X) \times (\pi_{n+1}(X, \text{pt}_X) =_{\text{Ab}} A))$$

Theorem 4 (Classifying 2-groups).  There is an equivalence of types:

$$2\text{Grp} \simeq ((G : \text{Grp}) \times (A : BG \rightarrow \text{Ab}) \times ((x : BG) \rightarrow_{\text{pt}} K(A(x), 3)))$$

Proof. Using the theorem of [45](#) and the foregoing propositions, we have the following chain of (natural) equivalences:

$$\begin{aligned} & (G : \text{Grp}) \times (A : BG \rightarrow \text{Ab}) \times ((x : BG) \rightarrow_{\text{pt}} K(A(x), 3)) \\ & \simeq (G : \text{Grp}) \times (A : BG \rightarrow \text{Ab}) \times ((x : BG) \rightarrow_{\text{pt}} T(K(A(x), 2))) \\ & \simeq (G : \text{Grp}) \times (A : BG \rightarrow \text{Ab}) \times (c : (x : BG) \rightarrow T(K(A(x), 2))) \times (c(\text{pt}_{BG}) = \text{pt}_{T(K(A(x), 2))}) \\ & \simeq (G : \text{Grp}) \times (A : BG \rightarrow \text{Ab}) \\ & \quad \times ((x : BG) \rightarrow (F : \mathcal{U}) \times \|F\| \times ((y : F) \rightarrow K(A(x), 2) \simeq_{\text{pt}} (F, y))) \times F(\text{pt}_{BG}) \\ & \simeq (G : \text{Grp}) \times (F : BG \rightarrow \mathcal{U}) \times F(\text{pt}_{BG}) \times (A : BG \rightarrow \text{Ab}) \\ & \quad \times ((x : BG) \rightarrow \|F(x)\| \times ((y : F(x)) \rightarrow 1\text{Conn}(F(x)) \times 2\text{Trunc}(F(x)) \times A = \pi_2(F(x), y))) \\ & \simeq (B : \mathcal{U}_{>0, \text{pt}}^{\leq 1}) \times (F : B \rightarrow \mathcal{U}_{>1}^{\leq 2}) \times F(\text{pt}_B) \\ & \quad \times ((x : B) \rightarrow (A : \text{Ab}) \times ((y : F(x)) \rightarrow A = \pi_2(F(x), y))) \end{aligned}$$

The last line is contractible, the expression following $((x : B) \rightarrow \dots$ being the type of fillers for the square

$$\begin{array}{ccc} F(x) & \longrightarrow & \text{Ab} \\ \downarrow & & \downarrow \\ 1 & \xlongequal{\quad} & 1 \end{array}$$

where we have that the left edge is 1-connected and the right edge is 1-truncated.

□

Proposition 32 (Identifying the action).  Given a 2-group A we can find a triple consisting of G , $H : BG \rightarrow \text{Ab}$, and $c : (u : BG) \rightarrow_{\text{pt}} K(H(u), 3)$.

G we know to be $\pi_1(A)$. Since A is connected, $B\pi_1(A) = \|A\|_1$. Because the type of abelian groups is 1-truncated, there is a canonical map $B\pi_1(A) \rightarrow \text{Ab}$, which factors the map $\lambda x. \pi_2(A, x) : A \rightarrow \text{Ab}$. This map is (essentially) H .

Proof. Recall how we constructed H , we first decomposed A into the sigma-type $(u : \|A\|_1) \times F(u)$ where $F(u) = \text{fib}_{|\cdot|_1}(u)$. $H : \|A\|_1 \rightarrow \text{Ab}$ was then defined so that it satisfied $(y : F(u)) \rightarrow H(u) = \pi_2(F(u), y)$. By the elimination rule for the one-truncation, it suffices to show that for each $x : A$, $H(|x|_1) = \pi_2(A, x)$.

So let's fix such an x , we can write $y := (x, \text{refl}) : F(|x|_1)$. Now, we have a fiber sequence in pointed types:

$$F(|x|_1) \longrightarrow A \longrightarrow \|A\|_1$$

Using the associated long-exact sequence we can show that $\pi_2(F(|x|_1), y) =$

$\pi_2(A, x)$.

Since H is defined to satisfy $(y : F(u)) \rightarrow H(u) = \pi_2(F(u), y)$, we know that $H(|x|_1) = \pi_2(F(|x|_1), y) = \pi_2(A, x)$. There is nothing else to show. This establishes the identity of the action from the classification theorem with the canonical one. $\square$

Proposition 33 (Identifying the invariant).  *The map $(u : BG) \rightarrow_{\text{pt}} K(H(u), 3)$ is equivalently a map sending a point u in BG to a $K(H(u), 2)$ -torsor (in the sense of Wärrn). The underlying type for this torsor is none other than $\text{fib}_{|\cdot|_1}(u)$.*

Proof. This follows fairly directly from the proof. After we have introduced $H(u)$, the unique abelian group such that $(y : F(u)) \rightarrow H(u) = \pi_2(F(u), y)$, we remember that $F(u)$ is 1-connected and 2-truncated, and we note that therefore the condition $(y : F(u)) \rightarrow H(u) = \pi_2(F(u), y)$ is equivalent to the condition $(y : F(u)) \rightarrow (F(u), y) \simeq_{\text{pt}} K(H(u), 2)$. It also follows from the connectedness that $F(u)$ is merely inhabited. This makes $F(u)$ into a $K(H(u), 2)$ -torsor. This is how we find a torsor for each u , its underlying type is $F(u)$. But remember (as we said before) $F(u) = \text{fib}_{|\cdot|_1}(u)$. There is nothing else to show. $\square$

2.4 Further discussion

Neither the statement nor the proof used any particular features of 2-groups, in fact we have the following:

Theorem 5 (Classifying n -groups).  Fix $1 < n < \infty$, there is an equivalence of types:

$$n\text{Grp} \simeq ((G : (n-1)\text{Grp}) \times (A : BG \rightarrow \text{Ab}) \times ((x : BG) \rightarrow_{\text{pt}} K(A(x), n+1)))$$

One way to think about these results is in terms of the Postnikov towers of truncated spaces. The cocycle is the map sending $x : \|BA\|_{n-1}$ to the fiber of the truncation map $BA \rightarrow \|BA\|_{n-1}$ over it. Up to homotopy, these cocycles are cohomology classes referred to classically as the k -invariants. The case for 2-groups (classically, crossed modules) is given already by Postnikov [\[33\]](#) as an application of his methods.

We can prove that $BA = (u : \|BA\|_{n-1}) \times (k(u) = \text{pt})$. This is independent of whether $\pi_n(BA)$ varies over $\|BA\|_{n-1}$.

As n -truncated types are classified by their Postnikov towers (see further on in this thesis for a definition), by Whitehead's principle, we effectively have another proof of the original result following from this remark. This also suggests why we do not expect a form of our Theorem [\[5\]](#) to extend to (untruncated) ∞ -groups in HoTT, but as the next section shows, there are some exceptional cases where our methods do extend.

2.5 Classifying some untruncated ∞ -groups

Definition 20. A pointed type X is called *central* if and only if the canonical map

$$(X \simeq X)_{(\text{id})} \rightarrow X$$

given by evaluation at the point, is an equivalence of pointed types.

Remark 3. If K is an Eilenberg-Mac Lane space of an abelian group, then K is central.

Theorem 6 (Buchholtz, Christensen, Flaten, and Rijke [12]). A connected higher group K is central if and only if the type of automorphisms $\text{Aut}_{\text{pt}}(K)$ is a set.

Corollary 4. If K is a central higher group, then so is BK .

Proposition 34 (Buchholtz [8]). For any connected higher group K , there is a fiber sequence

$$K \rightarrow B\text{Aut}_{\text{pt}}(K) \rightarrow B\text{Aut}(K)$$

(and the right-most map preserves the point).

Proof. Observe that $B\text{Aut}(K) \simeq_{\text{pt}} (X : \mathcal{U}) \times \|X \simeq K\|$ and $B\text{Aut}_{\text{pt}}(K) \simeq_{\text{pt}} (X : \mathcal{U}_{\text{pt}}) \times \|X \simeq_{\text{pt}} K\|$. The fiber of the forgetful map $B\text{Aut}_{\text{pt}}(K) \rightarrow B\text{Aut}(K)$

(which will preserve the point) is:

$$\begin{aligned}
& (X, x, p : \mathbf{BAut}_{\text{pt}}(K)) \times (X \simeq K) \\
& \simeq (X : \mathcal{U}) \times (X \simeq K) \times (x : K) \\
& \simeq K
\end{aligned}$$

since, given the equivalence in the first row, $\|X \simeq_{\text{pt}} K\|$ becomes an inhabited proposition, so we can omit p and reshuffle the rest of the data. $\square$

Corollary 5. *If K is simply connected, there is an equivalence:*

$$\pi_1(f) : \pi_1(\mathbf{BAut}_{\text{pt}}(K)) \simeq \pi_1(\mathbf{BAut}(K))$$

where f is the forgetful map from above.

Proof. There is a long exact sequence:

$$\dots \rightarrow \pi_1(K) \rightarrow \pi_1(\mathbf{BAut}_{\text{pt}}(K)) \rightarrow \pi_1(\mathbf{BAut}(K)) \rightarrow 0$$

and $\pi_1(K) = 0$ by assumption. $\square$

Corollary 6. *If K is simply connected and central, $\|\mathbf{BAut}(K)\|_1 \simeq_{\text{pt}} \mathbf{BAut}_{\text{pt}}(K)$.*

Proof. Using Theorem [6](#) and Corollary [5](#), the truncation of the forgetful map:

$$\|f\|_1 : \mathbf{BAut}_{\text{pt}}(K) \rightarrow \|\mathbf{BAut}(K)\|_1$$

(is pointed and) an equivalence by Whitehead's principle for truncated types. $\square$

Remark 4. *This implies that there is a map*

$$BAut(K) \rightarrow BAut_{pt}(K)$$

which will preserve the point.

Theorem 7 ([12]). *If K is a central higher group, then the connected cover*

$$BAut(K)\langle 1 \rangle \simeq (X : \mathcal{U}) \times \|X \simeq K\|_0$$

is a delooping of K .

Remark 5. *The connected cover is the fiber of the map*

$$BAut(K) \rightarrow \|BAut(K)\|_1$$

.

Proposition 35. *If K is a simply connected central higher group, and $H :$*

$BAut_{pt}(K)$, then

(i.) H is a central higher group.

(ii.) There exist equivalences such that the following diagram commutes:

$$\begin{array}{ccc} B\text{Aut}(K) & \longrightarrow & B\text{Aut}_{\text{pt}}(K) \\ \simeq \downarrow & & \downarrow \simeq \\ B\text{Aut}(H) & \longrightarrow & B\text{Aut}_{\text{pt}}(H) \end{array}$$

and the image of H under these equivalences is H again.

Now, if $t : B\text{Aut}(K) \rightarrow B\text{Aut}_{\text{pt}}(K)$ is the map from before, it follows from this and from Remark 5 that $\text{fib}_t(H) \simeq_{\text{pt}} BH$.

Proof. We tackle each item in turn

(i.) Using Definition 20, being central is a proposition for pointed types, and $\|H = K\|$ as pointed types, by assumption.

(ii.) If X is a type (respectively, pointed type) then, since $\|H = K\|$ as pointed types by assumption, the propositions $\|X \simeq K\|$ and $\|X \simeq H\|$ (respectively, $\|X \simeq_{\text{pt}} K\|$ and $\|X \simeq_{\text{pt}} H\|$) are equivalent. So we can map (X, p) in $B\text{Aut}(K)$ to (X, p') in $B\text{Aut}(H)$, and this mapping is an equivalence, likewise for (X, x, p) in $B\text{Aut}_{\text{pt}}(K)$. These equivalences clearly preserve H and will, moreover, commute with the forgetful maps like so:

$$\begin{array}{ccc} B\text{Aut}_{\text{pt}}(K) & \longrightarrow & B\text{Aut}(K) \\ \simeq \downarrow & & \downarrow \simeq \\ B\text{Aut}_{\text{pt}}(H) & \longrightarrow & B\text{Aut}(H) \end{array}$$

It follows that these next two diagrams commute, where the horizontal

maps are the pointed equivalences from before:

$$\begin{array}{ccc}
 B\text{Aut}_{\text{pt}}(K) & \longrightarrow & \|B\text{Aut}(K)\|_1 \\
 \simeq \downarrow & & \downarrow \simeq \\
 B\text{Aut}_{\text{pt}}(H) & \longrightarrow & \|B\text{Aut}(H)\|_1
 \end{array}
 \qquad
 \begin{array}{ccc}
 \|B\text{Aut}(K)\|_1 & \longrightarrow & B\text{Aut}_{\text{pt}}(K) \\
 \simeq \downarrow & & \downarrow \simeq \\
 \|B\text{Aut}(H)\|_1 & \longrightarrow & B\text{Aut}_{\text{pt}}(H)
 \end{array}$$

Then we have that the following diagram commutes:

$$\begin{array}{ccccc}
 B\text{Aut}(K) & \longrightarrow & \|B\text{Aut}(K)\|_1 & \xrightarrow{\simeq} & B\text{Aut}_{\text{pt}}(K) \\
 \simeq \downarrow & & \simeq \downarrow & & \downarrow \simeq \\
 B\text{Aut}(H) & \longrightarrow & \|B\text{Aut}(H)\|_1 & \xrightarrow{\simeq} & B\text{Aut}_{\text{pt}}(H)
 \end{array}$$

Which is what we sought to show.

We can deduce that the fiber over H of $B\text{Aut}(K) \rightarrow B\text{Aut}_{\text{pt}}(K)$ is equivalent to the fiber over H of $B\text{Aut}(H) \rightarrow B\text{Aut}_{\text{pt}}(H)$ (since the outermost vertical equivalences preserve H), but because the horizontal equivalences are pointed, and H is the point of $B\text{Aut}_{\text{pt}}(H)$, it follows that this fiber is the fiber over the point of $B\text{Aut}(H) \rightarrow \|B\text{Aut}(H)\|_1$, which is BH because H is central. $\square$

Theorem 8 (Myers higher Schreier theory [31]). *If G and K are higher groups, the type of higher groups A such that there exists a fiber sequence like so:*

$$BK \longrightarrow BA \longrightarrow BG$$

is equivalent to the type of pointed maps:

$$BG \longrightarrow {}_{\text{pt}}\text{BAut}(BK)$$

Theorem 9 (Really more like a corollary). *If G is a higher group, and K is a simply-connected central higher group, then the type of higher groups A such that there exists a fiber sequence like so:*

$$BK \longrightarrow BA \longrightarrow BG$$

is equivalent to the type of pairs of pointed maps:

$$(H : BG \rightarrow_{\text{pt}} \text{BAut}_{\text{pt}}(BK)) \times ((u : BG) \rightarrow_{\text{pt}} BH(u))$$

Proof. Using Myers higher Schreier theory, we see that the type is equivalent to $BG \rightarrow_{\text{pt}} \text{BAut}(BK)$.

Now, since BK is central, using previous results, we can decompose $\text{BAut}(BK)$ like so:

$$\text{BAut}(BK) \simeq (H : \text{BAut}_{\text{pt}}(BK)) \times \text{fib}_t(H) \simeq (H : \text{BAut}_{\text{pt}}(BK)) \times BH$$

and the result follows. □

We can recover the original and classical theorem for 2-groups using this machinery:

Proposition 36. *Recall that, if G is a 1-group, and K is an Eilenberg-Mac Lane space of an abelian group, then the type of higher groups A such that there exists a fiber sequence like so:*

$$BK \longrightarrow BA \longrightarrow BG$$

is equivalent to the type of pairs of pointed maps:

$$(H : BG \rightarrow_{\text{pt}} \text{BAut}_{\text{pt}}(BK)) \times ((u : BG) \rightarrow_{\text{pt}} BH(u))$$

From this we can deduce the expected 2-group classification theorem from before.

Proof. Following Myers, for fixed higher groups G and K , write $\text{EXT}(G, K)$ for those pointed types A that fit a fiber sequence as in the statement.

Then the type of 2-groups is equivalent to the type

$$(G : \text{Grp}) \times (H : \text{Ab}) \times \text{EXT}(G, BH)$$

Recall:

$$\text{BAut}_{\text{pt}}(BK) \simeq (X : \mathcal{U}_{\text{pt}}) \times \|X = BK\|$$

So the type of 2-groups is equivalent to the type

$$\begin{aligned}
& (G : \text{Grp}) \times (H : \text{Ab}) \times (X : BG \rightarrow \mathcal{U}_{\text{pt}}) \\
& \times (((u : BG) \rightarrow \|X(u) = B^2H\|) \times (X(\text{pt}) = B^2H)) \times ((u : BG) \rightarrow_{\text{pt}} BX(u)) \\
& \simeq_1 (G : \text{Grp}) \times (X : BG \rightarrow \mathcal{U}_{\text{pt}}) \times ((u : BG) \rightarrow 1\text{Conn}(X(u)) \times 2\text{Trunc}(X(u))) \\
& \times ((u : BG) \rightarrow_{\text{pt}} BX(u)) \\
& \simeq (G : \text{Grp}) \times (X : BG \rightarrow \mathcal{U}_{>1, \text{pt}}^{\leq 2}) \times ((u : BG) \rightarrow_{\text{pt}} BX(u)) \\
& \simeq_2 (G : \text{Grp}) \times (H : BG \rightarrow \text{Ab}) \times ((u : BG) \rightarrow_{\text{pt}} K(H(u), 3))
\end{aligned}$$

where, for the equivalence marked $\simeq_1$, we use the following facts (in the order given)

1. Since $(u : BG) \rightarrow \|X(u) = B^2H\|$, the type

$$(u : BG) \rightarrow 1\text{Conn}(X(u)) \times 2\text{Trunc}(X(u))$$

is an inhabited proposition, hence contractible, so can be added to the type we've written out.

2. Since BG is connected, if $X(\text{pt}) = B^2H$, the type

$$(u : BG) \rightarrow \|X(u) = B^2H\|$$

is an inhabited proposition, hence contractible, hence can be removed from the type we've written out.

3. The type $(H : \mathbf{Ab}) \times (B^2 H = X(\mathbf{pt}))$ is equivalent to the type

$$(H' : \mathcal{U}_{>1, \mathbf{pt}}^{\leq 2}) \times (H' = X(\mathbf{pt}))$$

which is contractible and hence can be removed from the expression.

On the line makes $\simeq_2$ we have used the equivalence between abelian groups and pointed, 1-connected 2-types, whose underlying function, we know, takes a group to its 2nd Eilenberg-Mac Lane space, whose unique delooping is the third Eilenberg-Mac Lane space. $\square$

Chapter 3

Hypercompletion

3.1 Introduction

Modalities were studied intensively in the setting of HoTT by Rijke, Shulman and Spitters [36]. We will briefly revisit some of their terminology and key theorems in the preliminaries section, alongside a few other established HoTT results and definitions that we will need going forward. After that we will spend some time on the concepts of countable products and countable inverse limits in HoTT. Some of that material has been formalized by the author. The two sections subsequent to that contain the main results of this work and are fully formalized. There is a description of the choice principles identified by Anel and Barton [3], their relationship to Postnikov towers, and a proof that a certain operation is a modality which presents the hypercomplete types. We conclude with an application to the setting of synthetic Stone duality, unformalized as of the time

of writing.

The question whether a hypercompletion operation can be defined in HoTT was posed in [36]. The work presented here (esp. the formalized portion) is a partial resolution to that question in the case where we are allowed to assume further classical axioms (variants of countable or dependent choice).

This chapter incorporates text from an article that is to appear in the TYPES 2025 post-proceedings [28].

3.2 Preliminaries

Back to modalities, drawing still on [36, esp. Section 3].

Example 2 (Open modality). *If P is a proposition, then there is a modality whose operation is*

$$\lambda X. (P \rightarrow X)$$

This modality is left-exact.

Definition 21 (Topological modality). *A modality is called topological if there is some (type-indexed) family of propositions P such that the modal types X (hence the whole modality) are characterized by the lifting properties for the squares*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ 1 & \xlongequal{\quad} & 1 \end{array}$$

Clearly each P is nullified by the modal operator.

Clearly the open modality is topological.

Proposition 37. *Topological modalities are lex.*

Proof. Corollary 3.12 in [36]. □

Definition 22. *A map f is called ∞ -connected if it is n -connected for every n .*

Proposition 38. *If A, B are types, $P : A \rightarrow \mathcal{U}$, and $Q : B \rightarrow \mathcal{U}$, and there are maps $f : A \rightarrow B$, and $g_a : P(a) \rightarrow Q(f(a))$ for all $a : A$, each of which is ∞ -connected, then $\lambda(a, p).(f(a), g_a(p)) : (a : A) \times P(a) \rightarrow (b : B) \times Q(b)$ is ∞ -connected.*

Proof. The fiber is $(a : A) \times (p : P(a)) \times (q : f(a) = b) \times (g_a(p) =^q p')$. This is equivalent to $(a : A) \times (q : f(a) = b) \times (p : P(a)) \times (g_a(p) =^q p')$ and that this is connected follows from the hypothesis. □

Definition 23. *A type X is called hypercomplete if it has right lifting against all ∞ -connected maps.*

Proposition 39. *Having left lifting against all maps with hypercomplete fibers characterizes ∞ -connected maps.*

Proof. We are trying to prove the bi-implication:

f is ∞ -connected $\Leftrightarrow f$ has left lifting against all maps with hypercomplete fibers

The left-to-right direction is immediate.

For the right-to-left direction, observe that the fibers of the map

$$\|f\|_{n+1} : \|X\|_{n+1} \rightarrow \|Y\|_{n+1}$$

will be $(n + 1)$ -truncated, hence hypercomplete. So the type of fillers for the square:

$$\begin{array}{ccc} X & \longrightarrow & \|X\|_{n+1} \\ \downarrow f & & \downarrow \|f\|_{n+1} \\ Y & \longrightarrow & \|Y\|_{n+1} \end{array}$$

is contractible, so there is some $u : Y \rightarrow \|X\|_{n+1}$, this map can be factorized like so:

$$\begin{array}{ccc} Y & \xrightarrow{u} & \|X\|_{n+1} \\ & \searrow & \nearrow u' \\ & \|Y\|_{n+1} & \end{array}$$

and it is straightforward to show that the map u' is an inverse for $\|f\|_{n+1}$. So the theorem follows using Corollary [1](#) □

Definition 24 (Hypercompletion Modality). *A hypercompletion modality if it exists is a modality whose modal types are precisely the hypercomplete types.*

In the corresponding orthogonal factorization system, the left maps will be precisely the ∞ -connected maps. The right maps will be those with hypercomplete fibers.

By Proposition [7](#) the hypercompletion is unique if it exists. Obviously there is some analogy between hypercompletion and truncation, but there are important

differences too.

Proposition 40 (Hypercompletion is lex). *If the hypercompletion modality exists, then the units must be ∞ -connected. This implies that a hypercompletion modality must be left exact.*

Proof. Suppose f is nullified by the modal operator, then it is ∞ -connected, in particular η is nullified by the modal operator. By Proposition [39](#)

The canonical map $A \times_B C \rightarrow \bigcirc A \times_{\bigcirc B} \bigcirc C$ is given by

$$\lambda(x, y, p) \cdot (\eta(x), \eta(y), \text{ap}_\eta(p))$$

and ap_η is ∞ -connected because η is by Proposition [15](#), as are the other two components. A triple of ∞ -connected maps is ∞ -connected by Proposition [38](#). There is nothing else to show. $\square$

Definition 25. *A modality is called cotopological if all propositions it nullifies were already contractible. This implies, the only modality that is both topological and cotopological is trivial.*

Proposition 41 (Hypercompletion is cotopological). *Any proposition nullified by hypercompletion is trivial.*

3.3 Countable limits

Definition 26. *A (countable) family is a function $\mathbb{N} \rightarrow \mathcal{U}$.*

Definition 27. A (countable) tower is a family A together with a dependent family of functions, $a : (n : \mathbb{N}) \rightarrow A_{n+1} \rightarrow A_n$.

Definition 28. The product of a family is the type of dependent functions

$$\prod A := (n : \mathbb{N}) \rightarrow A_n$$

Definition 29. The limit of a tower is the type

$$\lim(A, a) := \left(x : \prod A \right) \times ((n : \mathbb{N}) \rightarrow a_n(x(n+1)) = x(n))$$

Abusing notation a bit, if $x : \lim(A, a)$, we will often write x for the first component and q_x for the second component. We might suppress the map in the tower and just write $\lim A$ when it's clear from context.

Proposition 42.  If (A_{n+k}, a_{n+k}) is the tower obtained from A by shifting k places up, then $\lim(A_{n+k}, a_{n+k}) = \lim(A, a)$.

Proof. There is an obvious map $\lim(A, a) \rightarrow \lim(A_{n+k}, a_{n+k})$, which has contractible fibers as extensions of a term in the latter limit are essentially unique by definition. □

Definition 30. If X is a type, there is a constant tower given by X and the identity. Its limit is X .

Proposition 43. If (A, a) is a tower, X is a type, $f : A_0 \rightarrow X$, then there is another tower A' s.t. $A'_{n+1} = A_n$, and $A'_0 = X$. The limit of the new tower is $\lim(A, a)$ still.

Definition 31. A map of families is a family of maps.

Definition 32. Tower maps or ladders are families of maps $f_n : A_n \rightarrow B_n$ such that all the squares commute.

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} \\
 \downarrow & & \downarrow \\
 A_n & \xrightarrow{f_n} & B_n \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array}$$

Write $\text{hom}((A, a), (B, b))$ for the type of such ladders.

Abusing notation a bit, if $f : \text{hom}((A, a), (B, b))$ we will often write f for the first component and p_f for the second component.

Proposition 44.  If $(A, a), (B, b)$ are two towers then

$$((A, a) = (B, b)) \simeq ((f : \text{hom}((A, a), (B, b))) \times (n : \mathbb{N}) \rightarrow \text{isEquiv}(f_n))$$

Proof. We will apply the fundamental theorem of identity types (Theorem [1](#)).

The family $P := \lambda(B, b).((f : \text{hom}((A, a), (B, b))) \times (n : \mathbb{N}) \rightarrow \text{isEquiv}(f_n))$ is such that:

1. $P(A, a)$ is pointed, by the ladder of identity maps on the A_n .

2. We have the following chain of natural equivalences:

$$\begin{aligned}
& (B : \mathbb{N} \rightarrow \mathcal{U}) \times (b : (n : \mathbb{N}) \rightarrow B_{n+1} \rightarrow B_n) \\
& \quad \times ((f, e) : \mathbb{N} \rightarrow (A_n \simeq B_n)) \times (p_f : (n : \mathbb{N}) \rightarrow f_n \circ a_n = b_n \circ f_{n+1}) \\
& \simeq ((B : \mathbb{N} \rightarrow \mathcal{U}) \times (f : (n : \mathbb{N}) \rightarrow B_n \simeq A_n)) \\
& \quad \times ((b : (n : \mathbb{N}) \rightarrow B_{n+1} \rightarrow B_n) \times ((n : \mathbb{N}) \rightarrow b_n = f_n a_n f_{n+1}^{-1}))
\end{aligned}$$

which is contractible.

So we may apply the fundamental theorem of identity types and the proposition follows. $\square$

Proposition 45.  *Tower maps induce maps between limits which are equivalences if their respective components are.*

Proof. If $x = (x, q_x)$ is a term of type $\lim A$, then $\lambda n. f_n(x(n))$ is a term of type $\prod B$, and for each $n : \mathbb{N}$, $p_f(n) \cdot \text{ap}_{f_n}(q_x(n)) : b_n(f_{n+1}(x(n+1))) =_{B_n} f_n(x_n)$, so $\lambda n. f_n(x(n))$ forms a term of type $\lim B$.

By Proposition [44](#), there is an equivalence between the identity type of the type of towers, and the type of ladders of equivalences between the towers. So, we can appeal to the J-rule and the observation that the above construction on the identity ladder yields the identity map on the limit. $\square$

Proposition 46.  *If X is a type and for each $x : X$, we have a tower $P(x)$,*

then there is a tower whose types are

$$(x : X) \rightarrow P_n(x)$$

and whose arrows come from composition $(x : X) \rightarrow P_{n+1}(x) \rightarrow P_n(x)$

There is an equivalence of types

$$((x : X) \rightarrow \lim P(x)) \simeq \lim((x : X) \rightarrow P_n(x))$$

given by postcomposition with the projection maps from $\lim P(x)$.

Proof. This is a consequence of the so-called “type theoretic axiom of choice”

not to be confused with the axioms of choice we discuss below

$$\begin{aligned} (x : X) \rightarrow \lim P(x) &\simeq (x : X) \rightarrow \left((x' : \prod P(x)) \times a_n(x'(n+1)) = x'(n) \right) \\ &\simeq (x' : (x : X) \rightarrow \prod P(x)) \\ &\quad \times ((x : X) \rightarrow (n : \mathbb{N}) \rightarrow a_n(x'(x)(n+1)) = x'(x)(n)) \\ &\simeq (x' : \prod ((x : X) \rightarrow P(x))) \\ &\quad \times ((n : \mathbb{N}) \rightarrow a_n \circ x'(n+1) = x'(n)) \\ &\simeq \lim((x : X) \rightarrow P(x)) \end{aligned}$$

If f is a function in $(x : X) \rightarrow \lim P(x)$, then the equivalence on the second line sends it to $(\lambda x.f(x), \lambda x.q_{f(x)})$, which is composed with the projections through the equivalence on the third line as claimed. $\square$

Proposition 47.  A lifting problem

$$\begin{array}{ccc} Y & \longrightarrow & \lim A \\ \downarrow & & \\ Z & & \end{array}$$

has a solution if and only if each of the lifting problems

$$\begin{array}{ccc} Y & \longrightarrow & A_n \\ \downarrow & & \\ Z & & \end{array}$$

has a solution.

Proof. Immediate from Proposition 46.

□

Proposition 48. If $x, y : \prod A$, then $(x = y) \simeq \prod (x_n = y_n)$.

Proof. This is just function extensionality 42.

□

Proposition 49. If $x, y : \lim A$, then there is a tower whose types are $x_n =_{A_n} y_n$, and whose maps are $\lambda(p : x_{n+1} = y_{n+1}). q_x(n)^{-1} \cdot \text{ap}_{a_n}(p) \cdot q_y(n)$, and we have:

$$(x = y) \simeq \lim(x_n = y_n)$$

Proof. By well-known facts about the identity types of sum types, we have the following:

$$(x = y) \simeq (p : x = y) \times q_x =^p q_y$$

which by function extensionality becomes

$$(p : x = y) \times ((n : \mathbb{N}) \rightarrow q_x(n) =^p q_y(n))$$

Now using the J-rule, we can show that for any $(x, y : \lim A$, and) $p : x = y$

$$(q_x(n) =^p q_y(n)) \simeq (q_x(n)^{-1} \cdot \text{ap}_{a_n}(p_{n+1}) \cdot q_y(n) = p_n)$$

which implies that our original type is in fact equivalent to:

$$(p : \prod x_n = y_n)((n : \mathbb{N}) \rightarrow q_x(n)^{-1} \cdot \text{ap}_{a_n}(p_{n+1}) \cdot q_y(n) = p_n)$$

Which is precisely the limit of the tower defined before, which completes the proof. $\square$

Proposition 50. *If $A_n \rightarrow B_n \leftarrow C_n$ is a family of cospans, then $\prod A \rightarrow \prod B \leftarrow \prod C$ is a cospan of products, and $\prod A \times_{\prod B} \prod C \simeq \prod(A_n \times_{B_n} C_n)$.*

Proposition 51. *If $(A, a), (B, b), (C, c)$ are three families of maps, and there are ladders $f : A \rightarrow B$ and $g : C \rightarrow B$, then there is a tower $A_n \times_{B_n} C_n$, and the limit of this tower is equivalent to the pullback $\lim A \times_{\lim B} \lim C$*

Proof. If $(A, a), (B, b), (C, c)$ are three towers, and $f_n : A_n \rightarrow B_n$ and $g_n : C_n \rightarrow B_n$ are two tower-maps, then there are maps $A_{n+1} \times_{B_{n+1}} C_{n+1} \rightarrow A_n \times_{B_n} C_n$ given by $\lambda(x, y, p).(a_n(x), c_n(y), p_f(n)^{-1} \cdot \text{ap}_{b_n}(p) \cdot p_g(n))$. We will write H for the function in the third argument, for convenience.

Now,

$$\begin{aligned}\lim A \times_{\lim B} \lim C &\simeq (x : \lim A) \times (y : \lim C) \times f(x) = g(y) \\ &\simeq (x : \lim A) \times (y : \lim C) \times \lim(f_n(x_n) = g_n(y_n))\end{aligned}$$

Using Proposition [49](#), then the proposition follows by the type-theoretic axiom of choice. Observe that the term $\lambda n.f_n(x(n))$ of type $\lim B$ has for its second component the term $p_f(n) \cdot \text{ap}_{f_n}(q_x(n)) : b_n(f_{n+1}(x(n+1))) = f_n(x(n))$, similarly for $\lambda n.g_n(y(n))$

$$\begin{aligned}&(x : \lim A) \times (y : \lim C) \times \lim(f_n(x_n) = g_n(y_n)) \\ &\simeq (x : \prod A) \times (q_x : (n : \mathbb{N}) \rightarrow a(x(n+1)) = x(n)) \times \\ &\quad (y : \prod C) \times (q_y : (n : \mathbb{N}) \rightarrow c(y(n+1)) = y(n)) \times \\ &\quad (p : \prod f_n(x(n)) = g_n(y(n))) \times \\ &\quad ((n : \mathbb{N}) \rightarrow \text{ap}_{f_n}(q_x(n))^{-1} \cdot p_f(n)^{-1} \cdot p(n+1) \cdot p_g(n) \cdot \text{ap}_{g_n}(q_y(n)) = p(n)) \\ &\simeq (x : \prod A) \times (y : \prod C) \times (p : \prod f_n(x(n)) = g_n(y(n))) \times \\ &\quad ((n : \mathbb{N}) \rightarrow (q_x : a(x(n+1)) = x(n)) \times (q_y : c(y(n+1)) = y(n)) \times \\ &\quad (H(p(n+1)) =^{\text{ap}_f(q_x), \text{ap}_g(q_y)} p(n))) \\ &\simeq (x : \prod A) \times (y : \prod C) \times (p : \prod f_n = g_n) \times \\ &\quad ((n : \mathbb{N}) \rightarrow (a(x(n+1)), c(y(n+1)), H(p(n+1))) = (x(n), y(n), p(n))) \\ &\simeq \lim(\lambda n.A_n \times_{B_n} C_n, \lambda n.\lambda(x, y, p).(a(x), c(y), H(p)))\end{aligned}$$

Using the fact about pullbacks of products above, since the map of the triples is exactly as given here. $\square$

3.4 Choice principles

The following definitions and propositions appear in [3]. They have been formalized [27].

Definition 33. *Anel-Barton choice principles*

We say that countable choice of dimension $\leq d$ holds if whenever A, B are families of types, and $f : A_n \rightarrow B_n$ is a family of $(k + d)$ -connected maps between them, then the map $\prod f : \prod A \rightarrow \prod B$ is k -connected.

We say that dependent choice of dimension $\leq d$ holds if whenever $(A, a), (B, b)$ are towers, and $f : A_n \rightarrow B_n$ is a ladder of $(k + d)$ -connected maps between them, then the map $\lim f : \lim A \rightarrow \lim B$ is k -connected.

Proposition 52. *Countable choice of dimension $\leq d$ holds if and only if, whenever A is a family of $(k + d)$ -connected types, then $\prod A$ is k -connected.*

Proof. The fiber of f is the product of the fibers of the f_n by Proposition 50. In the other direction consider the constant family on 1. $\square$

Proposition 53. *Dependent choice of dimension $\leq d$ holds if and only if, whenever (A, a) is a family of $(k + d)$ -connected types, then $\lim(A, a)$ is k -connected.*

Proof. The fiber of f is the limit of the fibers of f_n using Proposition 51. In other direction consider the constant family on 1. $\square$

Proposition 54. *Dependent choice of dimension $\leq d$ implies countable choice of the same dimension*

Proof. $\prod A_n$ is the limit of $A_1 \times \cdots \times A_k$, the latter are $d + n$ connected if each of the A_i are. $\square$

Proposition 55.  *Countable choice of dimension $\leq d$ implies dependent choice of dimension $\leq d + 1$.*

Proof. Suppose the hypothesis holds, and (A, a) is a tower whose types are all $n + d + 1$ connected. Then $\prod A$ is $n + 1$ connected. For $x : \prod A$, each identity type $a_n(x(n + 1)) = x(n)$ is $n + d$ connected, because of Proposition 15, so $(n : \mathbb{N}) \rightarrow a_n(x(n + 1)) = x(n)$ is n -connected by countable choice. It follows that the limit, being a sum of n -connected types over an $n + 1$ connected type is itself n -connected. There is nothing else to show. $\square$

3.5 Postnikov effectiveness and hypercompletion

The Postnikov decomposition of a space was introduced by Postnikov 33. It was subsequently generalized to define a certain class of diagrams in an arbitrary higher topos 25. The definition can be internalized into HoTT like so:

Definition 34 (Postnikov tower). *A tower (A, a) is called a Postnikov tower if each A_n is n -truncated and each a_n is n -connected.*

Definition 35 (The Postnikov tower of a type). *If X is a type, the Postnikov tower of X is the tower whose types are $\|X\|_n$ and whose morphisms come from the universal property of truncations.*

Definition 36. *We write $\eta_X : X \rightarrow \lim \|X\|_n$ for the unique morphism from X to the limit of the Postnikov tower of X such that $\lambda x. \eta_X(x)_n = |\bullet|_n$ for all n .*

Definition 37 (Postnikov effectiveness). *If (A, a) is a Postnikov tower, then by the universal property, there exist maps $\|\lim(A, a)\|_n \rightarrow A_n$, these are maps ϵ_n such that $\epsilon_n \circ |\bullet|_n = \lambda x. x_n$ the projection map. In fact these maps form a ladder, and we say Postnikov effectiveness holds if they are all equivalences.*

Proposition 56 (Anel-Barton [3]).  *Anel-Barton dependent choice implies Postnikov effectiveness.*

Proof. Throughout we write $p_n : \lim(A, a) \rightarrow A_n$ for the projection map in an arbitrary tower.

We fix a natural number $d : \mathbb{N}$, and assume that Anel-Barton dependent choice of dimension $\leq d$ holds, we also fix a Postnikov tower (A, a) . We observe that we have a ladder of maps $f_n : \|\lim A_k\|_n \rightarrow A_n$, which is such that $f_n \circ |\bullet|_n^{\lim A_k} = p_n : \lim A_k \rightarrow A_n$.

Observe, it suffices to show that each p_n is n -connected, since then f_n factors an n -connected map through an n -connected map, implying that f_n itself is n -connected, which, since its source and target are both n -truncated, implies that f_n is an equivalence.

So, fix n , and find m such that $m \geq n + d$. Consider the restricted diagram A_{k+m} , and the constant diagram A_m . There is a ladder of maps between these diagrams $q_{n'} : A_{n'+m} \rightarrow A_m$, where $q_{n'} = a_m \circ \cdots \circ a_{n'+m-1}$. Each $q_{n'}$ is at least $m \geq n + d$ connected, so the map $\lim q_k : \lim A_{k+m} \rightarrow \lim_k A_m$ is at least n -connected. But $\lim A_{k+m}$ is none other than $\lim A_k$, $\lim_k A_m$ is none other than A_m , and $\lim q_k$ is none other than p_m . Hence, p_m is at least n -connected.

Finally, we observe that $p_n = a_n \circ \cdots \circ a_m \circ p_m$, and each of these maps are at least n -connected, so p_n is at least n -connected. Which completes the proof. $\square$

Proposition 57. *If Postnikov effectiveness holds then, by definition, there is a family of equivalences $\epsilon_n : \|\lim \|X\|_k\|_n \rightarrow \|X\|_n$ for any space X . As an equivalence, ϵ_n has an inverse ϵ_n^{-1} , and this inverse is none other than $\|\eta_X\|_n$.*

Proof. We write p_n for the projection map, as in the previous proof.

We have the following equations:

$$|\bullet|_n^X = p_n \circ \eta_X \quad p_n = \epsilon_n \circ |\bullet|_n^{\lim \|X\|_k}$$

by definition.

The second equation implies that

$$|\bullet|_n^{\lim \|X\|_k} = \epsilon_n^{-1} \circ p_n$$

Whence

$$\begin{aligned}\epsilon_n^{-1} \circ | \bullet |_n^X &= \epsilon_n^{-1} \circ p_n \circ \eta_X \\ &= | \bullet |_n^{\lim \|X\|_k} \circ \eta_X\end{aligned}$$

Which implies

$$\epsilon_n^{-1} = \|\eta_X\|_n$$

by the universal property of the n -truncation. □

Proposition 58.  *Postnikov effectiveness implies that the operation sending a type to the limit of its Postnikov tower is a modality. Which we call the Postnikov limit modality.*

Proof. We want to check that

$$\lambda f.f \circ \eta_X : ((x : \lim \|X\|_k) \rightarrow \lim \|P(x)\|_n) \rightarrow ((x : X) \rightarrow \lim \|P(\eta_X(x))\|_n)$$

is an equivalence for all $P : \lim \|X\|_k \rightarrow \mathcal{U}$, for all X .

Since limits commute with products (Proposition 46), we see it is enough to check that

$$\lambda f.f \circ \eta_X : ((x : \lim \|X\|_k) \rightarrow \|P(x)\|_n) \rightarrow ((x : X) \rightarrow \|P(\eta_X(x))\|_n)$$

for all n .

Since the type of n -types is an $(n+1)$ -type, the map $\lambda x.\|P(x)\|_n : \lim \|X\|_k \rightarrow$

$\mathcal{U}^{\leq n}$ factors through $|\bullet|_{n+1} : \lim \|X\|_k \rightarrow \|\lim \|X\|_k\|_{n+1}$, say by $\lambda x. Q(x) : \|\lim \|X\|_k\|_{n+1} \rightarrow \mathcal{U}^{\leq n}$, and since n -types are, in particular, $(n+1)$ -truncated, we see that the universal property implies it suffices to check that

$$\lambda f. f \circ \|\eta_X\|_{n+1} : ((x : \|\lim \|X\|_k\|_{n+1}) \rightarrow Q(x)) \rightarrow ((x : \|X\|_{n+1}) \rightarrow Q(\|\eta_X\|_{n+1}(x)))$$

is an equivalence for all $Q : \|\lim \|X\|_k\|_{n+1} \rightarrow \mathcal{U}^{\leq n}$

But this is trivial because $\|\eta_X\|_{n+1}$ is an equivalence by Proposition 57. $\square$

Proposition 59 (Postnikov limit modality is hypercompletion).   Assuming Postnikov effectiveness, the modal types for the Postnikov limit modality are precisely the hypercomplete types.

Proof. Assuming Postnikov effectiveness, the map $X \rightarrow \lim \|X\|_k$ is ∞ -connected. This we can see from the equation:

$$|\bullet|_{n+1}^{\lim \|X\|_k} \circ \eta_X = |\bullet|_{n+1}^X \circ p_{n+1}$$

Which shows that η factors an $(n+1)$ -connected map through an $(n+1)$ -connected map, so must itself be n -connected by Proposition 16.

If A_n is a Postnikov tower, then $\lim A_k$ is hypercomplete, this follows from the behavior of limits and lifting problems, see Proposition 47.

So, assuming Postnikov effectiveness, it follows that if X is hypercomplete, then η must be an equivalence – as it is an ∞ -connected map between hypercomplete types. And, if η is an equivalence, then X is hypercomplete by univalence

(as it is equivalent to a hypercomplete type). □

Thus, we have a hypercompletion modality under the assumption that (e.g.) countable choice holds.

3.6 Dependent choice in synthetic Stone duality

In the paper [16], a framework is developed for reasoning about the category of “light condensed anima”. The setting goes by the name of synthetic Stone duality (SSD) and has applications to condensed mathematics. They introduce a choice principle which we state below. We can prove Postnikov effectiveness from this choice principle, hence derive a characterization of the hypercompletion operation in their setting. The results in this section were arrived at in conversation with Hugo Moeneclaey and have not (yet) been formalized.

Definition 38. *SSD dependent choice*

If (A, a) is a tower of types and a_n are all (-1) -connected, then $p_0 : \lim A \rightarrow A_0$ is (-1) -connected.

Proposition 60. *SSD choice implies that if (A, a) is a tower of types, and the a_n are all n -connected, then $p_0 : \lim A \rightarrow A_0$ is n -connected.*

Proof. Using Proposition [15] it suffices to show that p_0 is (-1) -connected, and, for each $x, y : \lim A$, $\text{ap}_{p_0} : (x = y) \rightarrow (x_0 = y_0)$ is $(n - 1)$ -connected. The first is true by definition of SSD dependent choice.

Using Proposition 49, we know that $(x = y)$ is a limit of the tower

$$((x_n = y_n), \lambda p. q_x(n)^{-1} \cdot \text{ap}_{a_n}(p) \cdot q_y(n))$$

and ap_{p_0} is the projection map. Using induction, it suffices to assume that the result of the proposition holds for $(n - 1)$, and then it follows immediately that ap_{p_0} is $(n - 1)$ -connected.

There is nothing else to show. $\square$

Proposition 61. *SSD dependent choice implies dependent choice of dimension ≤ 1 .*

Proof. If (A, a) is a tower of $n + 1$ -connected types, then

$$\dots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 1$$

is a tower whose limit is just the same as $\lim(A, a)$, and whose morphisms are all now n -connected as they factor the $n + 1$ -connected maps into the unit like so: $A_{n+1} \rightarrow A_n \rightarrow 1$. So the map $\lim(A, a) \rightarrow 1$ is n -connected as well by SSD dependent choice. $\square$

Proposition 62. *Dependent choice of dimension ≤ 0 implies SSD dependent choice.*

Proof. Consider the ladder between the tower (A_{n+1}, a_{n+1}) , and the constant tower A_0 . By hypothesis all these maps are (-1) -connected, so the map of limits, which is precisely the projection, is (-1) -connected. $\square$

Proposition 63. *SSD dependent choice implies countable choice of dimension ≤ 0 .*

Proof. As before, $\prod A$ is limit of $1 \times A_1 \times \cdots \times A_k$, the projection maps to 1 are all of course n -connected if all the A_k are. So the map from the limit is also n -connected by foregoing. $\square$

Proposition 64. *SSD choice implies Postnikov effectiveness.*

Proof. Suppose (A, a) is a Postnikov tower.

By results in Section 3.3, $\lim A$ is also the limit of the towers, (A_{n+k}, a_{n+k}) . These maps, a_{n+k} , are all at least k -connected, whence it follows that $p_k : \lim A \rightarrow A_k$ is k -connected, and the remainder of the proof proceeds like in the other case. $\square$

Corollary 7. *In the setting of synthetic Stone duality, the hypercompletion operation exists and is given by $\lambda X. \lim \|X\|$.*

3.7 Conclusion

In this chapter, we have introduced some choice principles, derived from the work of Anel-Barton [3], and the synthetic Stone duality framework [16] and described their relationship to the form of the hypercompletion operator. Along the way we proved a number of useful propositions about countable inverse limits.

Chapter 4

Serre finiteness

All formalization discussed in this section is joint work with Reid Barton, Axel Ljungström and Anders Mörtberg [6]. Except for in the first section, definitions and theorems are mostly due to Barton and Campion.

4.1 Connected covers

A key result in traditional homotopy theory is the proof of Serre [38] that the homotopy groups of spheres are (amongst other things) finitely presented. Barton and Campion [5] have produced a novel proof of this result in homotopy type theory, and we have recently formalized the proof. One tool that is needed for the formalization is the notion of connected covers, due to Cartan and Serre [14] and G. W. Whitehead [47] independently. This section summarizes our work formalizing these tools in HoTT and proving the needed properties. Cubical

Agda code is available via GitHub: [\[32\]](#). More specific hyperlinks are provided for particular proposition and theorem statements.

Definition 39 (*n*-Connected Cover). *If X is a pointed space, then $X\langle n \rangle$ is defined to be the fiber of the pointed map $|\cdot|_n : X \rightarrow \|X\|_n$. That is, the connected cover is the pullback in the diagram:*

$$\begin{array}{ccc} X\langle n \rangle & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow |\cdot|_n \\ 1 & \longrightarrow & \|X\|_n \end{array}$$

It follows from the definition that the n -connected cover is n -connected. The type $X\langle n \rangle$ is also pointed, because $|\cdot|_n$ is pointed. In fact, we have the following:

Proposition 65. *$X\langle n \rangle$ is the universal pointed, n -connected space with a pointed map into X .*

Proof. By definition there is a pullback diagram:

$$\begin{array}{ccc} X\langle n \rangle & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow |\cdot|_n \\ 1 & \longrightarrow & \|X\|_n \end{array}$$

If Y is an n -connected space with a pointed map f into X , then $|\cdot|_n \circ f$ must factor through $\|Y\|_n$. But the latter space is contractible, so the map factors through 1. So we have a unique (up to homotopy) pointed map $Y \rightarrow X\langle n \rangle$ which composes with the top map in the square to return the original map $Y \rightarrow X$ by the universal property of the pullback. $\square$

Definition [39](#) and the universal property have appeared also in work of Buchholtz and Rijke [9](#) and Christensen and Scoccola [17](#).

Proposition 66 (Homotopy Groups of Connected Covers). 

$$\begin{aligned}\pi_k X\langle n \rangle &= 0, & \text{if } k \leq n \\ \pi_k X\langle n \rangle &= \pi_k X, & \text{otherwise.}\end{aligned}$$

Proof. Immediate using the universal property. □

Proposition 67.  If $m \geq n$, then $(X\langle n \rangle)\langle m \rangle = X\langle m \rangle$.

Proof. By universal properties, we have that the following diagram commutes:

$$\begin{array}{ccc} (X\langle n \rangle)\langle m \rangle & \xrightarrow{g_1} & X\langle n \rangle \\ u_2 \uparrow & \searrow u_1 & \downarrow f \\ X\langle m \rangle & \xrightarrow{g_2} & X \end{array} \quad \begin{array}{c} \nearrow h \end{array}$$

Now this implies $g_2 \circ u_1 \circ u_2 = f \circ g_1 \circ u_2 = g_2$ and $g_1 \circ u_2 \circ u_1 = h \circ u_1 = g_1$, but, by the universal property, these imply $u_1 \circ u_2 = \text{id}_{X\langle m \rangle}$ and $u_2 \circ u_1 = \text{id}_{(X\langle n \rangle)\langle m \rangle}$, and the proposition follows by univalence. □

Definition 40 (The Whitehead Tower). *The Whitehead tower of X is the following sequence of spaces and universal pointed maps:*

$$\dots \longrightarrow X\langle n+1 \rangle \longrightarrow X\langle n \rangle \longrightarrow \dots \longrightarrow X\langle 1 \rangle \longrightarrow X\langle 0 \rangle \longrightarrow X$$

4.1.1 Identifying the Fibers of the Whitehead Tower

Remark 6. *If the space X is pointed, then in the following diagram, with the top horizontal the Whitehead tower of X , and the bottom horizontal the Postnikov tower of X , the squares all commute and the verticals are all fiber sequences:*

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & X\langle n+1 \rangle & \longrightarrow & X\langle n \rangle & \longrightarrow & \dots \longrightarrow X\langle 1 \rangle \longrightarrow X\langle 0 \rangle \longrightarrow X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel \\
 \dots & \xlongequal{\quad} & X & \xlongequal{\quad} & X & \xlongequal{\quad} & \dots \xlongequal{\quad} X & \xlongequal{\quad} & X & \xlongequal{\quad} & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & \|X\|_{n+1} & \longrightarrow & \|X\|_n & \longrightarrow & \dots \longrightarrow \|X\|_1 & \longrightarrow & \|X\|_0 & \longrightarrow & 1
 \end{array}$$

Proof. Beginning with the fiber sequences:

$$\begin{array}{ccccccc}
 \dots & X\langle n+1 \rangle & X\langle n \rangle & \dots & X\langle 1 \rangle & X\langle 0 \rangle & X \\
 & \downarrow & \downarrow & & \downarrow & \downarrow & \parallel \\
 \dots & X & X & \dots & X & X & X \\
 & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
 \dots & \|X\|_{n+1} & \|X\|_n & \dots & \|X\|_1 & \|X\|_0 & 1
 \end{array}$$

We know by the respective universal properties that there must exist arrows between the connected covers and between the truncations making the diagram commute as in the statement.

But inspecting the proofs of the identities $(X\langle n \rangle)\langle n+1 \rangle = X\langle n+1 \rangle$ and $\| \|X\|_{n+1} \|_n = \|X\|_n$ reveals that these are precisely the universal maps as claimed.

Indeed, referring to the diagram in the proof of proposition [67](#) with $n+1$ in place of m we have that the unique arrow $X\langle n+1 \rangle \rightarrow X\langle n \rangle$ which makes the square commute is precisely $h = g_2 \circ u_2$. $\square$

Proposition 68. *All horizontals and verticals are fiber sequences in the following diagram:*

$$\begin{array}{ccccc}
 \Omega F & \longrightarrow & X\langle n+1 \rangle & \longrightarrow & X\langle n \rangle \\
 \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & X & \xlongequal{\quad} & X \\
 \downarrow & & \downarrow & & \downarrow \\
 F & \longrightarrow & \|X\|_{n+1} & \longrightarrow & \|X\|_n
 \end{array}$$

Proof. This is an application of the 3×3 lemma for pullbacks to the diagram:

$$\begin{array}{ccccc}
 X & \xlongequal{\quad} & X & \xleftarrow{x} & 1 \\
 \downarrow & & \downarrow & & \parallel \\
 \|X\|_{n+1} & \longrightarrow & \|X\|_n & \xleftarrow{|x|_n} & 1 \\
 \uparrow |x|_{n+1} & & \uparrow |x|_n & & \parallel \\
 1 & \xlongequal{\quad} & 1 & \xlongequal{\quad} & 1
 \end{array}$$

where x is the point of X . □

Proposition 69. *With F as in the diagram in proposition 68, $F = K(\pi_{n+1}X, n+1)$.*

Proof. F is a pointed n -connected, $(n+1)$ -truncated type, with the same homotopy group as X in its non-trivial dimension, and hence is the indicated Eilenberg-Mac Lane space.

To see that it is n -truncated, recall that n -truncated types are closed under dependent sums. To see that it is n -connected, recall that the universal map from a type to its n -truncation is always n -connected.

To see that the homotopy group of F agrees with that of X , observe that

we have an extended fiber sequence $\Omega\|X\|_n \rightarrow F \rightarrow \|X\|_{n+1} \rightarrow \|X\|_n$, by the Puppe lemma. So there is an exact sequence:

$$\pi_{n+1}(\Omega\|X\|_n) \rightarrow \pi_{n+1}F \rightarrow \pi_{n+1}\|X\|_{n+1} \rightarrow \pi_{n+1}\|X\|_n$$

and $\pi_{n+1}(\Omega\|X\|_n) = 0 = \pi_{n+1}\|X\|_n$, and $\pi_{n+1}\|X\|_{n+1} = \pi_{n+1}X$. □

Corollary 8 (The Fiber of the Whitehead Tower).  *With F as in proposition 68.* $\Omega F = K(\pi_{n+1}X, n)$.

4.2 Stably almost finite spaces

As was mentioned above, the formalization of the Barton-Campion proof of the Serre finiteness theorem was recently completed. A central, novel notion, key to the proof, and one of the signal contributions of the formalization, is that of a stably almost finite space.

In this section we are going to make use of a more general class of higher inductive types than we have so far, including joins, which are constructed from pushouts as we discussed above in the preliminaries.

Ljungström and Pujet [\[24\]](#) have developed the theory of finite CW-complexes in HoTT, we refer to their paper for a definition, and simply take as given that there is a notion of “being a CW-complex” defined for every type. In other words, for each type X there is another type $\text{isCW}(X)$ which refers to the notion of X being a (finite) CW-complex. One result we will need below is the following piece

of “obstruction theory”.

Proposition 70. *If X is a finite, n -dimensional CW-complex, and $f : Y \rightarrow Z$ is $(n - 1)$ -connected, then any map $X \rightarrow Z$ merely factors through f .*

We will also need to know that being a (finite, which is implicit henceforth) CW-complex is a proposition on types.

Proposition 71. *Being a CW-complex is a mere proposition for a fixed space.*

We need also to know the following

Proposition 72. *The pushout of a span of CW-complexes is again a CW-complex.*

Before discussing the proof of the Serre finiteness theorem, we state the following classification result about finitely presented groups.

Theorem 10 (Structure theorem for finitely presented abelian groups).  *If A is a finitely presented abelian group, then it is of the form*

$$\mathbb{Z}^k \oplus (\mathbb{Z}/k_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/k_m\mathbb{Z})$$

i.e. it is the direct sum of some free groups and some finite cyclic groups.

For a discussion of theorem [10](#) and its history see [41](#). The structure theorem was formalized in Cubical Agda by Axel Ljungström.

4.2.1 Stably almost finite spaces and related notions

Before defining stably almost finite spaces we introduce some auxiliary notions, as this was the order followed in the formalization, after Barton and Campion [5].

Definition 41. *A space X is called n -finite if there merely is an $(n-1)$ -connected map from some finite CW-complex into X .*

Proposition 73.  *Being n -finite is a mere proposition for a fixed space.*

The n -finite spaces satisfy some closure properties, and ultimately we are going to see that these closure properties are useful for proving related closure properties for the stably almost finite spaces, which are the spaces we will eventually use in the proof of the Serre finiteness theorem.

Proposition 74. *All finite CW-complexes are n -finite, for all n .*

Proof. This is immediate, using the identity map. □

Proposition 75.  *If X and Y are spaces, and X is $(n-1)$ -finite, and $f : X \rightarrow Y$ is $(n-2)$ -connected, then Y is also $(n-1)$ -finite.*

Proof. This is straightforward, since we are proving a proposition we may ignore the “merely” in the definition of the n -finite spaces, and so assume there is some $(n-2)$ -connected map $g : C \rightarrow X$ from a CW-complex into X , but then $f \circ g$ is an $(n-2)$ -connected map from a CW-complex into Y . There is nothing else to show. □

Proposition 76.  If X and Y are spaces, and Y is $(n - 1)$ -finite, and $f : X \rightarrow Y$ is $(n - 1)$ -connected, then X is also $(n - 1)$ -finite.

Proof. Again (and frequently in the propositions below) we are proving a proposition and so may omit the “merely” from the definition of n -finite spaces. So we can assume there is some $(n - 2)$ -connected map $g : C \rightarrow Y$ from a CW-complex into Y . This gives us a diagram like so

$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

So g will (merely) factor through f by Proposition [70](#), and, again, we can ignore the “(merely)” because we are proving a proposition. Because the map $C \rightarrow X$ factors an $(n - 1)$ -connected map by an $(n - 2)$ -connected map, it will itself be $(n - 2)$ -connected, proving that X is $(n - 1)$ -finite. $\square$

Proposition 77.  If X , Y and Z are spaces, and $X \rightarrow Y \rightarrow Z$ is a cofiber sequence and X is $(n - 1)$ -finite and Y is n -finite then Z is n -finite.

Proof. We may assume we have a diagram like so:

$$\begin{array}{ccccc} C & \longrightarrow & D & & \\ \downarrow f & & \downarrow g & & \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where f is $(n - 2)$ -connected and g is $(n - 1)$ -connected, applying Proposition [70](#) to find the map $C \rightarrow D$. Using Proposition [72](#) the cofiber of the map $C \rightarrow D$

is a CW-complex, C' , and by the universal property of the pushout, there is a map $C' \rightarrow Z$, making the following diagram commute

$$\begin{array}{ccccc} C & \longrightarrow & D & \longrightarrow & C' \\ \downarrow f & & \downarrow g & & \downarrow h \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

and the map $C' \rightarrow Z$ will be $(n - 1)$ -connected using Proposition 22. There is nothing else to show. $\square$

Definition 42. A space X is called stably n -finite if there merely is some m such that $\text{Susp}^m X$ is $(m + n)$ -finite.

Proposition 78.  Being stably n -finite is a mere proposition for a fixed space.

Proposition 79. All finite CW-complexes are stably n -finite for all n .

Proof. Using Proposition 72, we see that the suspension of a CW-complex is always again a CW-complex. $\square$

Proposition 80. If X and Y are spaces, and X is stably $(n - 1)$ -finite and $f : X \rightarrow Y$ is $(n - 2)$ -connected, then Y is stably $(n - 1)$ -finite.

Proof. By Proposition 22, if f is $(n - 2)$ -connected, then $\text{Susp}^m f : \text{Susp}^m X \rightarrow \text{Susp}^m Y$ must be $(m + n - 2)$ -connected. So we can pick m so that $\text{Susp}^m X$ is $(m + n - 1)$ -finite, and then appeal to Proposition 75. $\square$

The proof of the following proposition is quite similar.

Proposition 81. *If X and Y are spaces, and Y is stably $(n - 1)$ -finite, and $f : X \rightarrow Y$ is $(n - 1)$ -connected, then X is stably $(n - 1)$ -finite*

The proof of the following proposition is immediate.

Proposition 82.  *If X is a space, then $\text{Susp}X$ is stably n -finite if and only if X is stably $(n - 1)$ -finite.*

Proposition 83.  *If X , Y and Z are spaces and $X \rightarrow Y \rightarrow Z$ is a cofiber sequence, then:*

1. *if X is stably $(n - 1)$ -finite and Y is stably n -finite, then Z is stably n -finite,*
2. *if Y is stably $(n - 1)$ -finite and Z is stably n -finite, then X is stably $(n - 1)$ -finite, and*
3. *if X is stably $(n - 1)$ -finite and Z is stably $(n - 1)$ -finite then Y is stably $(n - 1)$ -finite*

Proof. The proof is by induction, using the co-Puppe lemma (Proposition 24) and Proposition 77 □

Proposition 84.  *If X and Y are spaces, and X is m -connected for some $m \geq 0$, and stably 0-finite, and Y is $(m' - 2)$ -connected and stably n -finite, then $X * Y$ is stably $\min[m', n + m + 2]$ -finite.*

Proof. There merely exists an (-1) -connected map from $1 \rightarrow X$, because X is (at least) path-connected. Since we are seeking a proof of a mere proposition,

we can apply elimination. It suffices to prove the claim from the following data:

$$\begin{array}{ccc} 1 & & C \\ \downarrow & & \downarrow \\ X & & \text{Susp}^k Y \end{array}$$

Since 1 is arbitrarily highly connected, we can apply Proposition 23 to show that the join of the maps in the diagram is $(\min[k + m' - 3, n + m + k - 1] + 2)$ -connected. It follows that $X * \text{Susp}^k Y \simeq \text{Susp}^k(X * Y)$ is at least $\min[m' + k - 1, n + m + k + 1]$ -finite, which, since $k \geq 0$, is what we want to show. $\square$

Proposition 85.  Let X be $(m - 2)$ -connected and stably n -finite with $(m - 1) \leq n$, and let Y be $(m' - 2)$ -connected and stably n' -finite, then $X * Y$ is stably $\min[m' + n, m + n']$ -finite.

Proof. Much as above, it is sufficient to prove the claim using the following diagram:

$$\begin{array}{ccc} C & & D \\ \downarrow & & \downarrow \\ \text{Susp}^k X & & \text{Susp}^k Y \end{array}$$

(Notice we choose k large enough that we can deal with X and Y uniformly). C must be at least $(k + m - 2)$ -connected, and so the join of the maps is at least $(\min[2k + n' + m - 3, 2k + n + m' - 3] + 2)$ -connected. It follows that $X * Y$ must be at least $\min[m + n', m' + n]$ -finite as required. $\square$

Definition 43. A space is called stably almost finite (SAF) if it is stably n -finite for every n .

Proposition 86.  Being stably almost finite is a proposition for a fixed space.

Proposition 87.  All finite CW-complexes are stably almost finite.

Proof. This is precisely the content of Proposition [79](#). □

Proposition 88. If X and Y are spaces and $f : X \rightarrow Y$ is ∞ -connected, then X is stably almost finite if and only if Y is.

Proof. We can combine Proposition [80](#) and Proposition [81](#), and the proof follows immediately. □

Proposition 89. If X , Y and Z are spaces and $X \rightarrow Y \rightarrow Z$ is a cofiber sequence, then if any two are stably almost finite, then so is the third.

Proof. This follows from the three parts of Proposition [83](#). □

Proposition 90. If X and Y are both stably almost finite, then so is $X * Y$.

Proof. We can combine [84](#) and [85](#), and the proof follows immediately. □

4.2.2 Applications to the Serre finiteness theorem

Proposition 91.  For all $n \geq 1$, if X is stably almost finite, and n -connected, then $\pi_{n+1}X$ is a finitely presented abelian group.

This depends on the weak Hurewicz theorem, which has been formalized in Cubical Agda by Ljungström and Pujet [\[24\]](#).

Proposition 92 (Ganea's theorem). Let $F \rightarrow E \rightarrow B$ be a fiber sequence, write E' for the cofiber of $F \rightarrow E$, then there is a map $E' \rightarrow B$, and we can form a new fiber sequence $F' \rightarrow E' \rightarrow B$, the fiber $F' = \text{Susp}(F \wedge \Omega B) \simeq \text{Susp}F \wedge \Omega B \simeq F * \Omega B$.

This had been formalized, but not by us, in Cubical Agda [2]. To see that $\text{Susp}F \wedge \Omega B \simeq F * \Omega B$ we can apply the 3×3 lemma (Proposition 21) to the diagram

$$\begin{array}{ccccc}
 1 & \longleftarrow & X & \longrightarrow & X \\
 \uparrow & & \uparrow & & \uparrow \\
 1 & \longleftarrow & X \vee Y & \longrightarrow & X \times Y \\
 \downarrow & & \downarrow & & \downarrow \\
 1 & \longleftarrow & Y & \longrightarrow & Y
 \end{array}$$

We can iterate this construction to obtain successive fiber sequences $F'' \rightarrow E'' \rightarrow B$, $F''' \rightarrow E''' \rightarrow B, \dots$ where $F^{(n+1)} = \text{Susp}^n F * (\Omega B)^{*(n+1)}$, where we write $(\Omega B)^{*2}$ for $\Omega B * \Omega B$, and likewise for larger n . This fiber becomes increasingly connected. If B is connected and we set $E^{(\infty)}$ to be the colimit of the $E^{(n)}$, then $E^{(\infty)} \rightarrow B$ is in fact an equivalence. Otherwise, $E^{(\infty)}$ is the image of the map $E \rightarrow B$, per Rijke [35].

Proposition 93.  If ΩB is stably almost finite, then so is B .

Proof. Consider the succession of fiber sequences

$$\begin{array}{ccccccc}
 \Omega B & \longrightarrow & (\Omega B)^{*2} & \longrightarrow & \text{Susp}(\Omega B)^{*3} & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & E' & \longrightarrow & E'' & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 B & \xlongequal{\quad} & B & \xlongequal{\quad} & B & \xlongequal{\quad} & \dots
 \end{array}$$

By induction, using the closure of SAF spaces under cofibers, each $E^{(n)}$ is SAF, then because the connectivity of the lower vertical maps is increasing, B must

be SAF. □

Proposition 94.  *If B is stably almost finite and simply connected, then so is ΩB .*

Proof. Consider the succession of fiber sequences from above.

We proceed recursively: the map $\Omega B \rightarrow 1$ is (-1) -connected, so ΩB is stably 0-finite by proposition [81](#). Now suppose ΩB is stably n -finite, then each $\Sigma^{(n-1)}(\Omega B)^{*n}$ is stably at least $(n+2)$ -finite: we have to consider two cases, one where ΩB is stably 0-finite, and the other where ΩB is stably at least 1-finite. Observe that ΩB is 0-connected, so we can apply Proposition [84](#) with $m = 0$, $m' = 2$, and $n = 1$. In the other case ΩB is still 0-connected, but now is stably n -finite for $n \geq 1$, so we can apply Proposition [85](#) with $m = m' = 2$.

Furthermore, $E^{(i)}$ is eventually stably $(n+2)$ -finite because the map $E^{(i)} \rightarrow B$ is eventually sufficiently highly connected for large i , now using proposition [83](#) we have that $E^{(i-1)}, \dots, E'', E' \simeq \text{Susp} \Omega B$ are too. So ΩB is stably $(n+1)$ -finite. □

Proposition 95.  *If $F \rightarrow E \rightarrow B$ is a fiber sequence, F and B are stably almost finite, and B is simply connected, then E is stably almost finite.*

Proof. Again we construct a succession of fiber sequences

$$\begin{array}{ccccccc}
 F & \longrightarrow & F * \Omega B & \longrightarrow & \text{Susp}(F * (\Omega B)^{*2}) & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 E & \longrightarrow & E' & \longrightarrow & E'' & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 B & \xlongequal{\quad} & B & \xlongequal{\quad} & B & \xlongequal{\quad} & \dots
 \end{array}$$

Using proposition 94 and our assumptions, we have that each $\text{Susp}^{(n-1)} F * (\Omega B)^{*n}$ is stably almost finite. Then because $E^{(i)} \rightarrow B$ is highly connected for large i , and B is SAF, $E^{(i)}$ is eventually stably n -finite, then we can use proposition 83 to show that $E^{(i-1)}, \dots, E'', E', E$ are stably n -finite, so SAF, since n was arbitrary. $\square$

Proposition 96.  If A is a finitely presented abelian group, then $K(A, n)$ is stably almost finite for any $n \geq 1$.

Proof. Using theorem 10 we know that A has the form:

$$\mathbb{Z}^k \oplus (\mathbb{Z}/k_1\mathbb{Z}) \oplus \dots \oplus (\mathbb{Z}/k_m\mathbb{Z})$$

So $K(A, n) = K(\mathbb{Z}^k, n) \times K(\mathbb{Z}/k_1\mathbb{Z}, n) \times \dots \times K(\mathbb{Z}/k_m\mathbb{Z}, n)$.

Using proposition 93, we may reduce to the case where $n = 1$. $K(\mathbb{Z}^k, 1) = S^1 \times \dots \times S^1$, which is a finite CW-complex, and hence is stably almost finite. Meanwhile, $\Omega K(\mathbb{Z}/k_1\mathbb{Z}, 1) = \mathbb{Z}/k_1\mathbb{Z}$ which is a finite set, hence stably almost finite. Now we can apply the closure properties of SAF spaces and there is

nothing else to show. □

The proof depends on proposition [93](#) and on the decomposition of finitely presented abelian groups into direct sums of free abelian groups and finite abelian groups.

With all of these propositions together, we are now able to give the proof of the Serre finiteness theorem.

Proof of the Serre finiteness theorem. 

We can prove directly that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_k(S^1) = 0$ for $k > 1$.

So, we may assume we are dealing with a simply connected space X , furthermore, we can assume X is stably almost finite by proposition [87](#). Then, by the discussion in section 1, we have the following fiber sequences:

$$K(\pi_{n+1}X, n) \rightarrow X\langle n+1 \rangle \rightarrow X\langle n \rangle$$

for all $n \geq 1$. We show by induction that each $\pi_{n+1}X$ is finitely presented, and each $X\langle n \rangle$ is stably almost finite.

Observe that the base case is true, since X is simply connected, which implies $X\langle 1 \rangle = X$ and π_2X is finitely presented by proposition [91](#). Now suppose $X\langle n \rangle$ is stably almost finite, and $\pi_{n+1}X$ is finitely presented, then, using the fiber sequence above, together with propositions [95](#) and [96](#), we see that $X\langle n+1 \rangle$ must be stably almost finite. Then, $\pi_{n+2}X$ is the first non-trivial homotopy group of $X\langle n+1 \rangle$, a stably almost finite and $(n+1)$ -connected space, so $\pi_{n+2}X$ is finitely presented. That proves the induction step.

We have established that each homotopy group of X is finitely presented, which completes the proof. $\square$

After 'black-boxing' some of the propositions above, it was relatively straightforward to formalize this argument in Cubical Agda using a `mutual` block, this was the first part of the project completed back in 2022.

Chapter 5

Conclusion

In this thesis we've presented three studies on synthetic homotopy theory; the first on higher group theory, where we recovered and even generalized the celebrated classification of 2-groups due in different forms to Mac Lane-Whitehead and Sính; the second on modalities, where we showed that, assuming any one of a family of variants of the axiom of choice, we can construct a hypercompletion modality in synthetic homotopy theory; the third where we summarized some contributions, made jointly with our several collaborators, to the formalization of the Serre finiteness theorem. In each case the Postnikov tower made an appearance, and indeed the first two studies covered ground that Postnikov himself trod in his thesis of 1951. Hopefully, after perusing this document, the reader recognizes the utility of this method of decomposing a space or a type, and perhaps they also see some intrinsic interest in a few of the novel results we have presented.

We'll take the liberty in the conclusion of briefly sketching a couple of other

proofs, which slightly tighten the connection between two of our chapters, using the Whitehead tower to shed some more light on the shape of the hypercompletion modality. After that we will point out a few loose ends from this work.

5.1 Connections

Proposition 97. *If X is a pointed space, then the limit of the Whitehead tower of X is the fiber of the Postnikov unit map η over the point of X .*

Proof. This follows immediately from our Definition 39, and the fact that limits commute with pullbacks, which is the content of Proposition 51. $\square$

Proposition 98. *The hypercompletion modality (exists and) is presented by the Postnikov modality and its unit if and only if for any space X and any point x in X , the limit of the Whitehead tower of the pointed space (X, x) is ∞ -connected*

Proof. Using Proposition 97 we see that the limit of the Whitehead tower of (X, x) is ∞ -connected for all x in X , if and only if the unit map is ∞ -connected as a map. But if the unit map is ∞ -connected, then we see immediately from the proof of Proposition 58, that the Postnikov modality exists. Then by inspecting the proof of Proposition 59 we see that in this case where the unit is ∞ -connected, that modality is the hypercompletion.

Conversely, we know that if the hypercompletion is presented by the Postnikov modality, then the units must be ∞ -connected. $\square$

5.2 Loose ends

We list some questions left open at the end of our studies. We have tried to group questions by the corresponding chapter in the thesis, but otherwise nothing is implied by the ordering.

Question 1. *To identify the Sính invariant of some non-trivial 2-groups.*

For product 2-groups, of the form $K(G, 1) \times K(H, 2)$, the Sính invariant is 0, the trivial cocycle, but otherwise the author would struggle to say anything concrete. It is not even immediately apparent what methods could be appropriate for computing the cocycle, or its image as a cohomology class, since (for example) the constructive nature of HoTT rules out standard projective resolution methods in group cohomology [46].

Question 2. *To classify the (hypercomplete) $H\mathbb{Z}$ -modules using their cohomological invariants.*

These represent a simple case of the kind of higher structures accessible to the techniques of ‘stable homotopy theory’. The development of stable homotopy theory has previously not been subject to much attention in HoTT (exceptions include the theses of van Doorn [43] and Riley [37]) but with the introduction of techniques from simplicial type theory, Weinberger, Buchholtz and Gratzner has delivered an important advance [19]. A particular goal would be to show, as in [1, 15] that the $H\mathbb{Z}$ -module spectra are “GEMs” – i.e. special ‘products’ of distinguished spectra called ‘Eilenberg-Mac Lane spectra’. Together with Buchholtz in

2024 we worked out a proof for the particular case of 2-dimensional $H\mathbb{Z}$ -module spectra, and it would be desirable to extend this to the more general case.

Question 3. *Possibility of another proof of the classification theorems for 2-groups using the Postnikov tower explicitly.*

The reader will notice that we did not give, in the previous section, any further remarks on the relationship between our first study and the other two. It seems like there should be another proof of (a suitable generalization of) the Sính classification theorem, which explicitly uses the Postnikov tower, but, with two proofs in hand, we have not yet taken the time to work one out.

Question 4. *Possibility of a hypercompletion modality which is neither the Postnikov modality nor the trivial modality.*

In the work of Morel and Voevodsky [29] (considerably pre-dating HoTT) they give an example of a hypercomplete ∞ -topos where not every type is the limit of its Postnikov tower. Given that HoTT has semantics in ∞ -toposes [39] it seems reasonable to take this as a clue that it is consistent for there to exist a hypercompletion operation which is not presented by the Postnikov modality. However in this case it happens to coincide with the trivial modality. Whether these options exhaust the possible shapes of the hypercompletion operation or not is (to the best of my knowledge) an open problem.

Question 5. *To formalize the more refined Serre finiteness theorem.*

Serre's original theorem was in fact stronger than what has been obtained so far, it was already known to topologists that $\pi_n(S^n) = \mathbb{Z}$ for all $n > 0$, and that

$\pi_k(S^n) = 0$ if $k < n$, what Serre proved was that if $k > n$, then $\pi_k(S^n)$ is *finite* if n is odd, and if n is even then $\pi_k(S^n)$ is finite *unless* $k = 2n - 1$ in which case $\pi_k(S^n)$ is the direct sum of $\mathbb{Z}$ and a finite group [38].

Also in HoTT it has been proven that $\pi_n(S^n) = \mathbb{Z}$ for all $n > 0$ [22], and that $\pi_k(S^n) = 0$ if $k < n$. What remains is the cases where $k > n$. A partial proof was sketched by Caldwell at a conference talk at CMU in 2021 [13]. The main gap in the proof appears to have been the need to first prove that the groups in question are all finitely presented. Now that this fact has been formalized, we expect to be able to formalize Caldwell's proof of the main theorem. This is a natural follow-up to work mentioned above. It would represent another important advance for the HoTT community, and doubtless provide the basis for much future work in type theory on the homotopy groups of the spheres. Caldwell's proof makes use of two key tools: the Gysin sequence for cohomology, which has been formalized in Cubical Agda by Ljungström and Mörtberg [23], and the technique of 'rationalization', which has not been formalized in Cubical Agda. For the first tool, we see that this project would represent a vital application of existing machinery, for the second we have no doubt that the formalization of this tool would be of independent interest.

Chapter 6

Bibliography

- [1] J. F. Adams. *Stable Homotopy and Generalised Homology*. Chicago Lectures in Mathematics. University of Chicago Press, 1974.
- [2] The Agda Community. *Cubical Agda library*. GitHub repository. URL: <https://github.com/agda/cubical/>.
- [3] M. Anel and R. Barton. “Choice axioms and Postnikov completeness”. 2024. URL: <https://arxiv.org/abs/2403.19772>.
- [4] S. Awodey and M. A. Warren. “Homotopy theoretic models of identity types”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 146.45 (2009).
- [5] R. Barton and T. Campion. “A synthetic approach to finite presentability of homotopy groups”. Preprint. 2026. URL: <https://github.com/CMU-HoTT/serre-finiteness/blob/main/Paper/barton-campion.pdf>.

- [6] R. Barton et al. “A computer formalisation of the Serre finiteness theorem”. In: *LICS 2026: Forty-First Annual Symposium on Logic in Computer Science*. Ed. by C. Faggian and J.-P. Katoen. 2026.
- [7] M. Bezem et al. *Symmetry*. <https://github.com/UniMath/SymmetryBook>. Commit: d252eb1. 2026.
- [8] U. Buchholtz. *Non-abelian cohomology (Groups, Torsors, Gerbes, Bands & all that)*. 2019. URL: <https://youtu.be/eB6HwGLASJI>.
- [9] U. Buchholtz and E. Rijke. “The long exact sequence of homotopy n -groups”. In: *Mathematical Structures in Computer Science* 33.8 (2023).
- [10] U. Buchholtz, F. van Doorn, and E. Rijke. “Higher groups in homotopy type theory”. In: *LICS '18: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. 2018.
- [11] U. Buchholtz and M. Williams. “Synthetic Homotopy Theory with HoTT/UF”. Unpublished lecture notes. 2024.
- [12] U. Buchholtz et al. “Central H-spaces and banded types”. In: *Journal of Pure and Applied Algebra* 229.6 (2025).
- [13] W. Caldwell. “Sphere Bundles and Characteristic Classes”. Talk for the CMU HoTT Graduate Student Workshop. 2021. URL: <https://cmu-hott.github.io/grad-workshop-2021-files/slides-caldwell.pdf>.
- [14] H. Cartan and J.-P. Serre. “Espaces fibrés et groupes d'homotopie”. In: *Comptes Rendus de l'Académie des Sciences*. 1952.

- [15] C. Casacuberta and J. J. Gutiérrez. “Homotopical localizations of module spectra”. In: *Transactions of the American Mathematical Society* 357.7 (2005).
- [16] F. Cherubini et al. “A Foundation for Synthetic Stone Duality”. In: *30th International Conference on Types for Proofs and Programs (TYPES 2024)*. Ed. by R. E. Møgelberg and B. van den Berg. 2025.
- [17] J. D. Christensen and L. Scoccola. “The Hurewicz theorem in homotopy type theory”. In: *Algebraic & Geometric Topology* 23.5 (2023).
- [18] E. Finster and D. R. Licata. “Eilenberg-MacLane spaces in homotopy type theory”. In: *CSL–LICS ’14: Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. 2014.
- [19] D. Gratzer, J. Weinberger, and U. Buchholtz. “Cocompleteness in simplicial homotopy type theory”. In: *TYPES 2025: The 31st International Conference on Types for Proofs and Programs*. Ed. by F. N. Forsberg. 2025.
- [20] P. Hart and O. Milner. “Classifying 2-groups in homotopy type theory”. In: *LICS 2026: Forty-First Annual Symposium on Logic in Computer Science*. Ed. by C. Faggian and J.-P. Katoen. 2026.
- [21] L. Leclerc. “Cohomology in Homotopy Type Theory”. Internship Report. 2022.

- [22] D. R. Licata and G. Brunerie. “ $\pi_n(S^n)$ in Homotopy Type Theory”. In: *Certified Programs and Proofs*. Ed. by G. Gonthier and M. Norrish. 2013.
- [23] A. Ljungström and A. Mörtberg. “Formalizing $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda”. In: *38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. 2023.
- [24] A. Ljungström and L. Pujet. “Cellular methods in homotopy type theory”. 2025. URL: <https://pujet.fr/pdf/cellular.pdf>.
- [25] J. Lurie. *Higher Topos Theory*. Princeton University Press, 2009.
- [26] S. MacLane and J. H. C. Whitehead. “On the 3-type of a complex”. In: *Proceedings of the National Academy of Sciences of the United States of America* 36 (1950).
- [27] O. Milner. GitHub Repository. URL: <https://github.com/owen-milner/choicepostnikov>.
- [28] O. Milner. “Choice principles and hypercompletion in HoTT”. In: *31st International Conference on Types for Proofs and Programs (TYPES 2025)*. Ed. by F. N. Forsberg and J. McKinna. 2026.
- [29] F. Morel and V. Voevodsky. “ A^1 -homotopy theory of schemes”. In: *Publications mathématiques de l’I.H.É.S* 90 (1999).
- [30] B. A. Munson and I. Volić. *Cubical homotopy theory*. New Mathematical Monographs. Cambridge University Press, 2015.

- [31] D. J. Myers. “Symmetry, Geometry and Modality”. PhD thesis. Johns Hopkins University, 2022.
- [32] CMU-HoTT Organization. GitHub repository. URL: <https://github.com/CMU-HoTT/serre-finiteness>.
- [33] M. M. Postnikov. “Investigations in the homotopy theory of continuous mappings”. In: *American Mathematical Society Translations*. 2nd ser. 7 (1957).
- [34] E. Rijke. *Introduction to Homotopy Type Theory*. Cambridge University Press, 2026.
- [35] E. Rijke. “The join construction”. 2017. URL: <https://arxiv.org/abs/1701.07538>.
- [36] E. Rijke, M. Shulman, and B. Spitters. “Modalities in homotopy type theory”. In: *Logical Methods in Computer Science* 16.1 (2020).
- [37] M. Riley. “A Bunched Homotopy Type Theory for Synthetic Stable Homotopy Theory”. PhD thesis. Wesleyan University, 2022.
- [38] J.-P. Serre. “Homologie singulière des espaces fibrés”. In: *Annals of Mathematics* 54.3 (1951).
- [39] M. Shulman. “All $(\infty, 1)$ -toposes have strict univalent universes”. 2019. URL: <https://arxiv.org/abs/1904.07004>.
- [40] H. X. Sinh. “Gr-catégories”. PhD thesis. Institut Pédagogique no. 2 de Hanoi, 1975.

- [41] J. Stillwell. *Classical Topology and Combinatorial Group Theory*. Second Edition. Graduate Texts in Mathematics. Springer, 1993.
- [42] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations for Mathematics*. Institute for Advanced Study, 2013. URL: <https://homotopytypetheory.org/book/>.
- [43] F. van Doorn. “On the formalization of higher inductive types and synthetic homotopy theory”. PhD thesis. Carnegie Mellon University, 2018.
- [44] V. Voevodsky. “Univalent foundations project”. Modified NSF grant application. 2010.
- [45] D. Wörn. “Eilenberg-MacLane spaces and stabilisation in homotopy type theory”. In: *Journal of Homotopy and Related Structures* 18 (2023).
- [46] C. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.
- [47] G. W. Whitehead. “Fiber spaces and the Eilenberg homology groups”. In: *Proceedings of the National Academy of Sciences*. Vol. 38. 5. 1952.