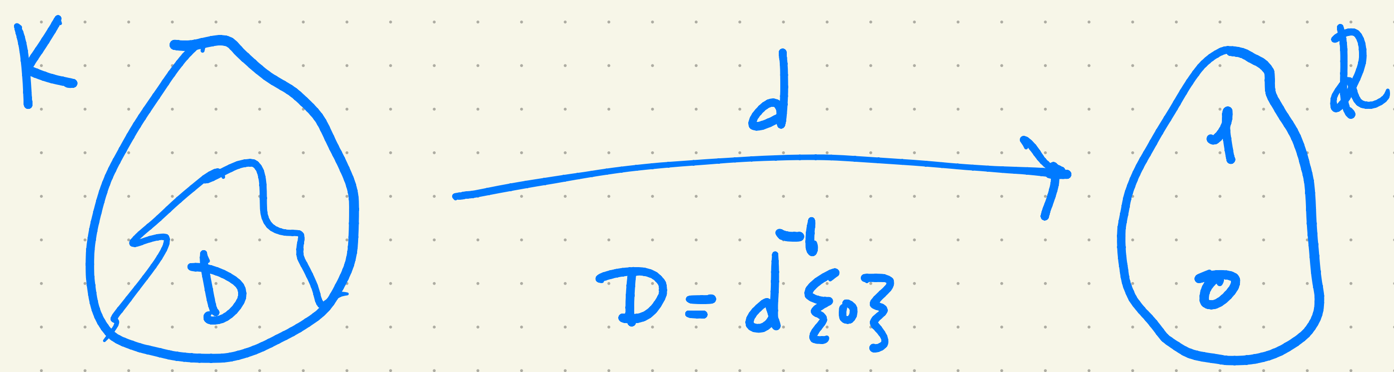
# Kripke Models of $\lambda$-Calculus

CMU - SS'23

For the propositional logic PPC we had semantics in CC posets, of which one special kind was the downsets in any poset $K$,

$$\mathrm{Down}(K) \cong \mathrm{Pos}(K, 2) \cong 2^K$$




This gave rise to Kripke semantics for PPC by:

$$\frac{[\![-]\!] : PPC \longrightarrow 2^K}{\Vdash : PPC \times K \longrightarrow 2}$$

where:

$$[\![\varphi]\!]_k = 0 \quad \in 2$$

$$\Longleftrightarrow$$

$$k \Vdash \varphi \, .$$

- Then functoriality & ccc of the model

$$[\![-]\!] : \mathrm{PPC} \longrightarrow \mathbf{2}^{K}$$

gave rise to the Kripke conditions:

  - $j \leq k \Vdash \varphi \rightarrow j \Vdash \varphi$
  - $k \Vdash \varphi \Rightarrow \psi$ iff $\forall_{k \leq \ell}.\ \ell \Vdash \varphi$ implies $\ell \Vdash \psi$
  - etc.

- For completeness we took:

$$K = \mathcal{C}_{\mathrm{PPC}}^{op}$$

$$[\![-]\!] = \downarrow : \mathcal{C}_{\mathrm{PPC}} \longrightarrow \mathbf{2}^{\mathcal{C}_{\mathrm{PPC}}^{op}}$$

which was just the Yoneda embedding, factored through $\mathbf{2} \hookrightarrow \mathrm{Set}$:

$$\begin{tikzcd}
\mathcal{C}_{\mathrm{PPC}} \arrow[r, "\downarrow"] \arrow[dr, hook, bend right, "y"'] & \mathbf{2}^{\mathcal{C}_{\mathrm{PPC}}^{op}} \arrow[d, hook] & \mathbf{2} \arrow[d, hook] \\
 & \mathrm{Set}^{\mathcal{C}_{\mathrm{PPC}}^{op}} & \mathrm{Set}
\end{tikzcd}$$

Then we generalized from Propositions to Types.

$$\begin{array}{ccc} \text{STT} & & 1, \times, \to \\ & & (0, +) \\ | & & \\ \text{PL} & & \top, \wedge, \Rightarrow \\ & & (\bot, \vee) \end{array}$$

The (syntactic) classifying category $\mathcal{C}_{\mathbb{T}}$ of a $\lambda$-theory $\mathbb{T}$ is a CCC, and a model of $\mathbb{T}$ in any CCC $\mathcal{C}$ is a CC functor,

$$M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C} .$$

We have a completeness theorem w/resp. to variable models, i.e. those of the form

$$M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Set}^{\mathbb{C}} .$$

as a common generalization of:

- functorial semantics of algebraic theories,
- Kripke semantics of propositional logic.

The completeness theorem is again proved "by Yoneda":

$$\mathcal{C} = \mathcal{C}_{\mathbb{T}}^{op}$$

$$\mathcal{U} = y : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Set}^{\mathcal{C}_{\mathbb{T}}^{op}}.$$

Then we obtain:

1) $y$ is a model, since it's cc.

2) $\mathbb{T} \vdash (\cdot \vdash a : A)$ "inhabitation"

$\Leftrightarrow \ \mathcal{U} \Vdash a : A$ "PAT provability"

$\Leftrightarrow \ 1 \longrightarrow [\![A]\!]^M$ "pointedness" in all $M$

by fullness of $y$ (and universality)

3) $\mathbb{T} \vdash u = v : A$ provably

$\Leftrightarrow \ \mathcal{U} \models u = v : A$ in $\mathcal{U}$

$\Leftrightarrow \ M \models u = v : A$ in all $M$

by faithfullness of $y$ (and universality)

**Thm** (Scott 1980) (Presheaf Completeness)

For any $\lambda$-theory $\mathbb{T}$ we have:

(1) $\mathbb{T} \vdash t : T \iff \mathcal{M} \models T$
f. all $\mathbb{C}$ and all $\mathbb{T}$ models $\mathcal{M}$ in $\hat{\mathbb{C}}$

(2) $\mathbb{T} \vdash s = t : E \iff \mathcal{M} \models s = t$
f. all $\mathbb{C}$ and all $\mathbb{T}$ models $\mathcal{M}$ in $\hat{\mathbb{C}}$.

Pf:

1. Build the syntactic CCC $\mathcal{C}_{\mathbb{T}}$, consisting of types & terms, mod equ's.
2. $\mathcal{C}_{\mathbb{T}}$ has a canonical model $\mathcal{U}$, consisting of the basic types & terms.
3. $\mathcal{U}$ is generic, in the sense:

$$\mathbb{T} \vdash t : T \iff \mathcal{U} \models T$$

$$\mathbb{T} \vdash s = t : E \iff \mathcal{U} \models s = t .$$

4. $\mathcal{C}_{\mathbb{T}}$ is the free CCC on a $\mathbb{T}$-model:

$\forall \mathcal{M}\ \exists!\, \mathcal{M}^{\#}$ :

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{\mathcal{M}^{\#}} & \mathcal{C} \quad \text{CCC} \\ \mathcal{U} & \longmapsto & \mathcal{M} \end{array} .$$

Next we need the following generalization of the main lemma from PL for $\downarrow : P \to \hat{P}$.

Lemma For any small cat $\mathbb{C}$, the cat

$$\hat{\mathbb{C}} = \mathrm{Set}^{\mathbb{C}^{op}}$$

of presheaves on $\mathbb{C}$ is cartesian closed, and the Yoneda embedding

$$y : \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$$

preserves any CCC structure in $\mathbb{C}$.

pf. For $P, Q \in \hat{\mathbb{C}}$ what should $Q^P$ be?

$$\begin{aligned} (Q^P)c &\cong \hat{\mathbb{C}}(yc, Q^P) && \text{Yoneda} \\ &\cong \hat{\mathbb{C}}(yc \times P, Q) && \text{CCC} \end{aligned}$$

So let $Q^P := \hat{\mathbb{C}}(y(-) \times P, Q)$. ✓

Given $c, d \in \hat{\mathbb{C}}$,

$$\begin{aligned} y(d^c) &= \mathbb{C}(-, d^c) && \text{def} \\ &= \mathbb{C}(- \times c, d) && \text{CCC} \\ &\cong \hat{\mathbb{C}}(y(- \times c), yd) && \text{Yoneda} \\ &\cong \hat{\mathbb{C}}(y(-) \times yc, yd) && \text{UMP of } \times \\ &= (yd)^{yc}. \ ✓ && \text{def} \end{aligned}$$

To finish the proof of the thm:

if $\mathbb{T} \vdash t : T$, then $\mathcal{U} \models T$, namely

(*) $\qquad [\![t]\!] : 1 \longrightarrow [\![T]\!]$ in $\mathcal{C}_{\mathbb{T}}$.

Given any $\mathbb{T}$ model $\mathcal{M}$ in any $\hat{\mathbb{C}}$,

since $\mathcal{C}_{\mathbb{T}}$ is free on $\mathcal{U}$ we have:

$$\mathcal{C}_{\mathbb{T}} \xrightarrow{\mathcal{M}^{\#}} \mathcal{C}$$

$$\mathcal{U} \longmapsto \mathcal{M}.$$

So from (*) we obtain:

$$\begin{tikzcd}
\mathcal{M}^{\#} 1 \arrow[r, "\mathcal{M}^{\#} {[\![t]\!]}"] \arrow[d, phantom, "\cong" description] & \mathcal{M}^{\#} [\![T]\!] \arrow[d, phantom, "\cong" description] \\
1 \arrow[r] & [\![T]\!]^{\mathcal{M}}
\end{tikzcd}$$

where the $\cong$'s are because $\mathcal{M}^{\#}$ is ccc.

Thus indeed:

$$\mathcal{M} \models T.$$

Conversely, if $\mathcal{M} \models T$ for all $\mathcal{M}$, then in particular $\mathcal{U} \models T$.

Whence:

$$\mathbb{T} \vdash T,$$

since $\mathcal{U}$ is generic.

(1) ✓

(2) %

The result can now be specialized from general cats $\mathbb{C}$ to posets $K$:

**Thm** (Kripke Completeness of $\lambda$-Calculus)
For any $\lambda$-theory $\mathbb{T}$ we have:

(1) $\mathbb{T} \vdash t : T \iff \mathcal{M} \models T$
f. all posets $K$ and $\mathbb{T}$-models $\mathcal{M}$ in $\hat{K}$

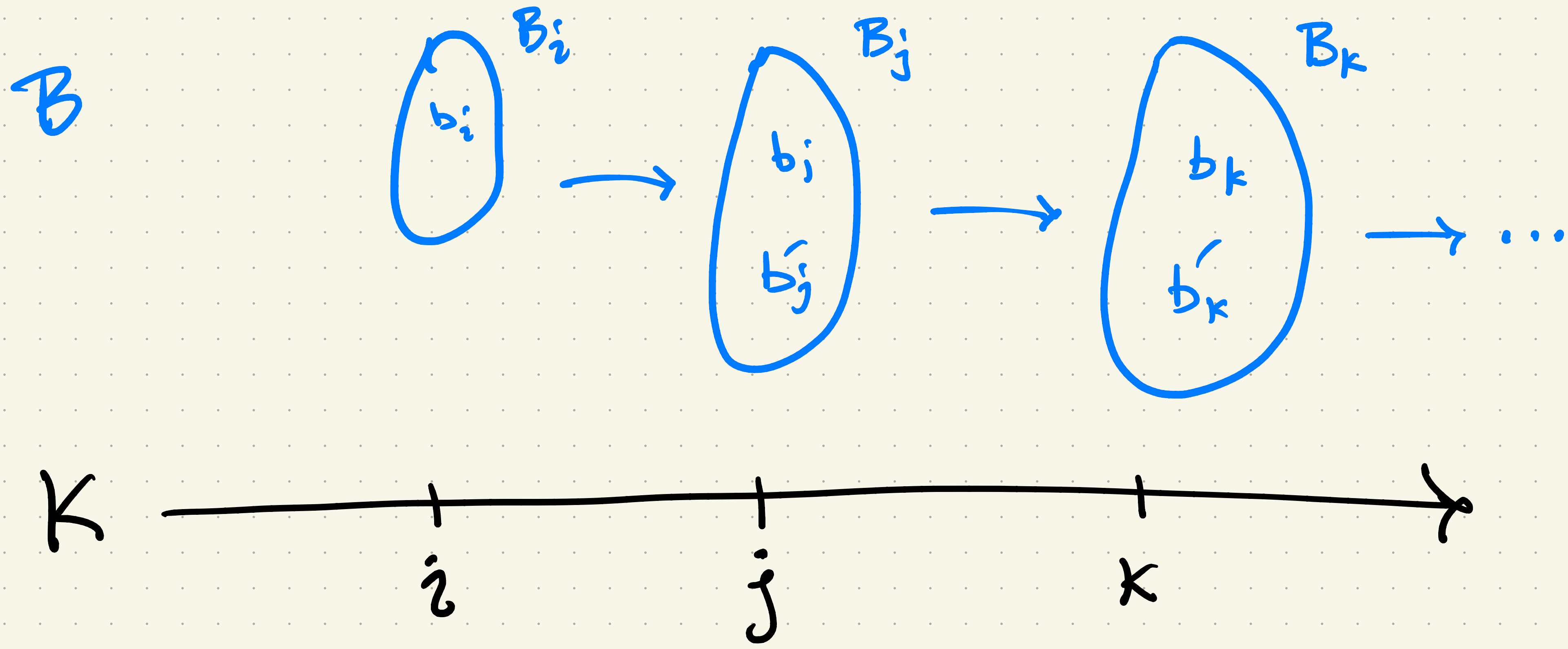
(2) $\mathbb{T} \vdash s = t : E \iff \mathcal{M} \models s = t$
f. all posets $K$ and $\mathbb{T}$-models $\mathcal{M}$ in $\hat{K}$.

The proof* uses a theorem from topos theory (due to Joyal & Tierney) to move from $\hat{\mathbb{C}}$ to $\hat{K}_{\mathbb{C}}$ for a poset $K_{\mathbb{C}}$.

Remark: One can also go between "Scott style" $j \in [\![A]\!]$ and "Kripke style" $j \Vdash A$ for $\lambda$-theories (see AGH 2021).

* In (AR 2011).

# What is a Kripke model of the λ-calculus?




$k \Vdash b : B$ f.a. $k \in K$

"pointwise inhabitation"

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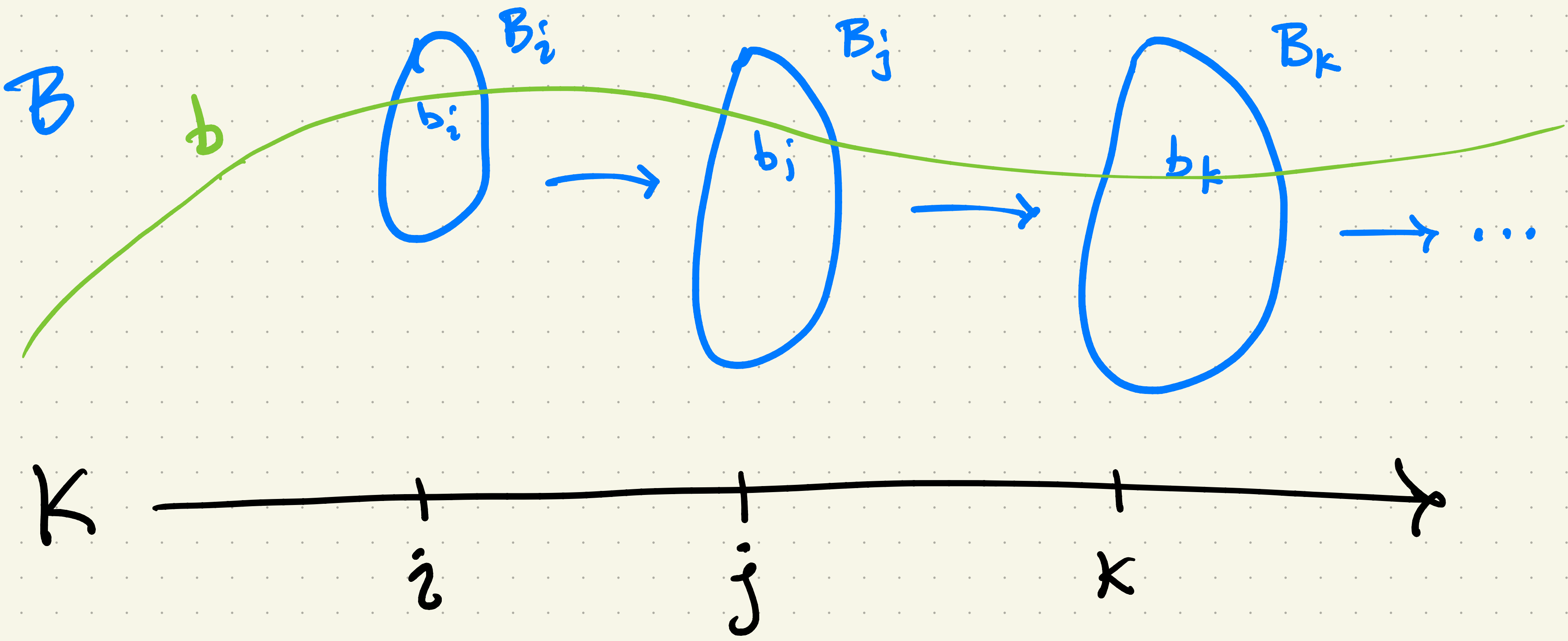


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Versus

$K \Vdash b : B$

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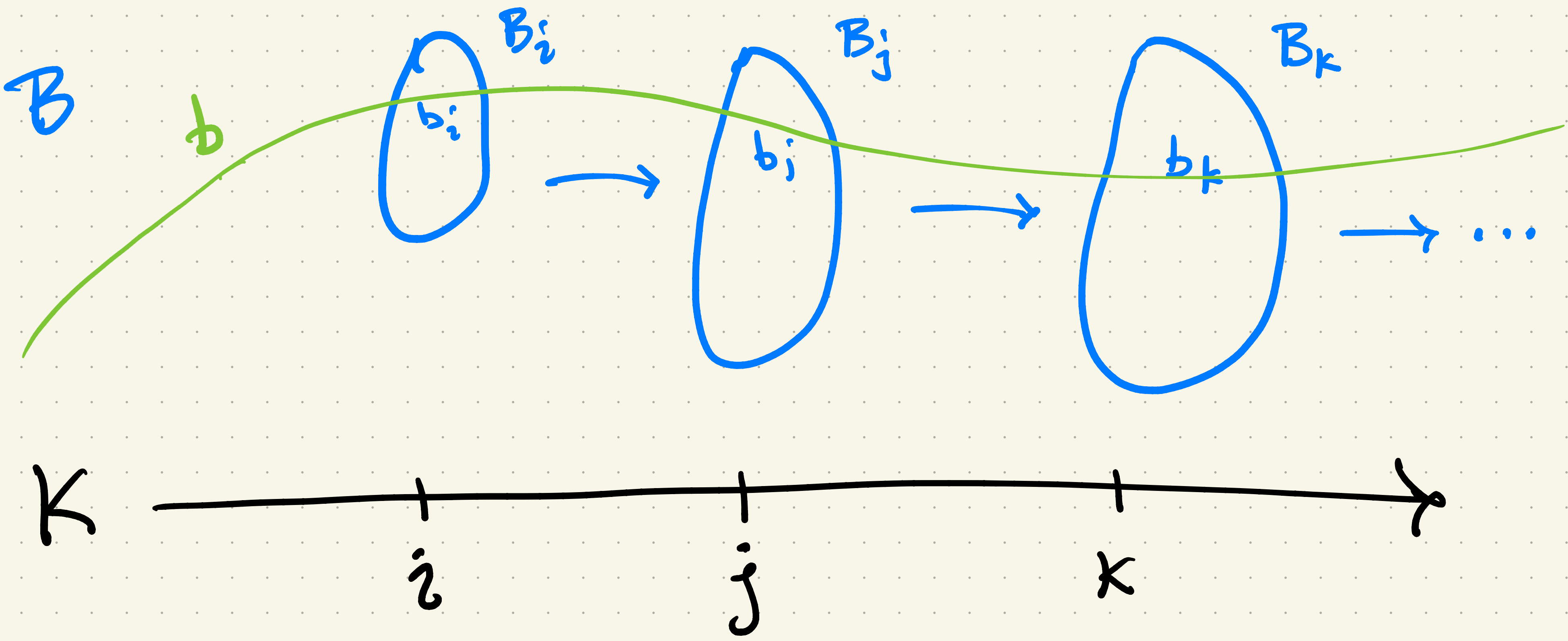


$k \Vdash b : B$ f.a. $k \in K$

"pointwise inhabitation"

Versus

$K \Vdash b : B$

"global inhabitation"

$\iff$ f.all $k : K$, there's a $b_k : B_k$

$k \Vdash b_k : B_k$

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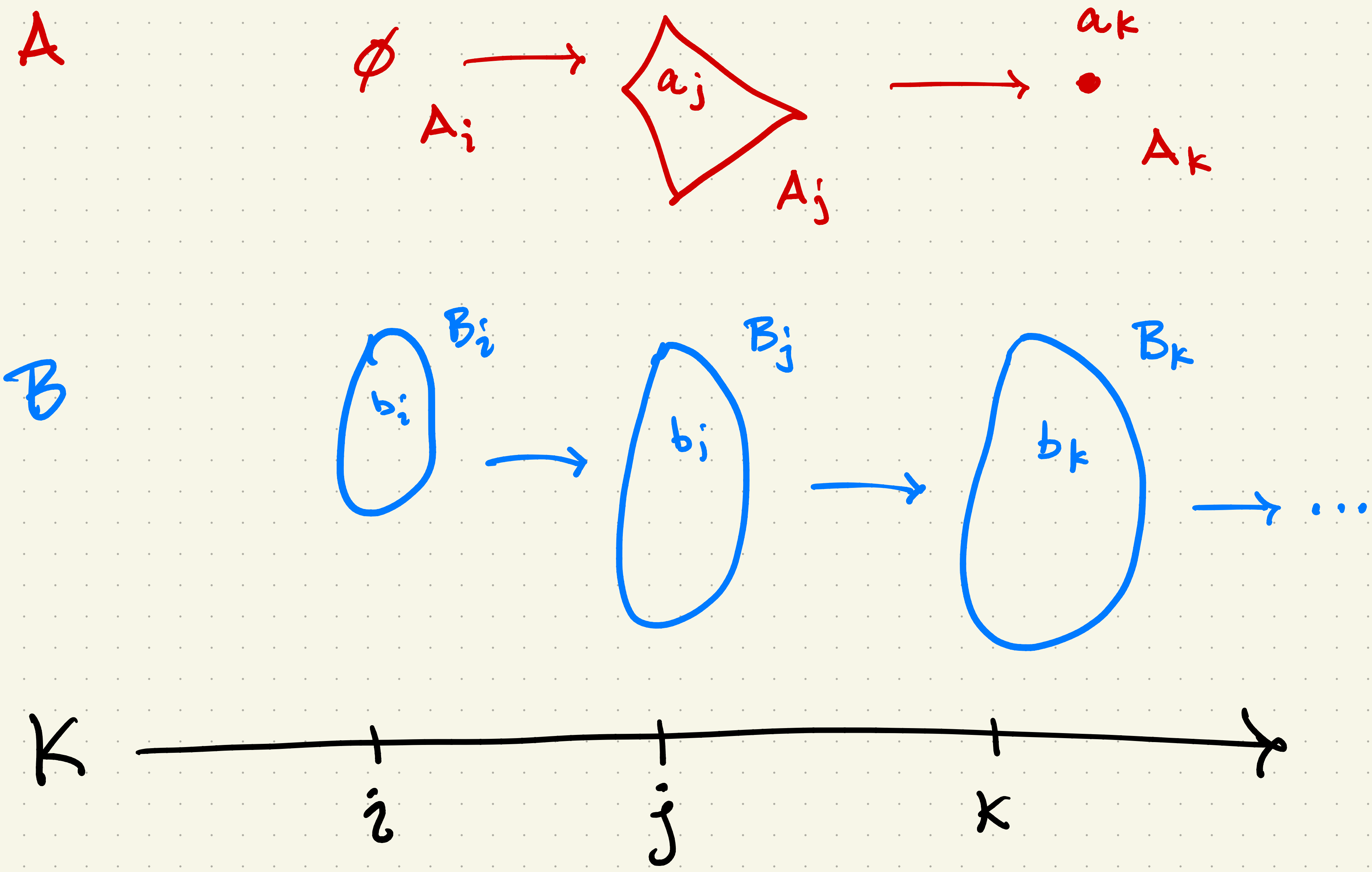


$$j \Vdash a : A \Rightarrow k \Vdash a : A$$

$$K \nVdash a : A$$

A not inhabited

$$\Rightarrow \quad \mathcal{F} \nvdash A$$

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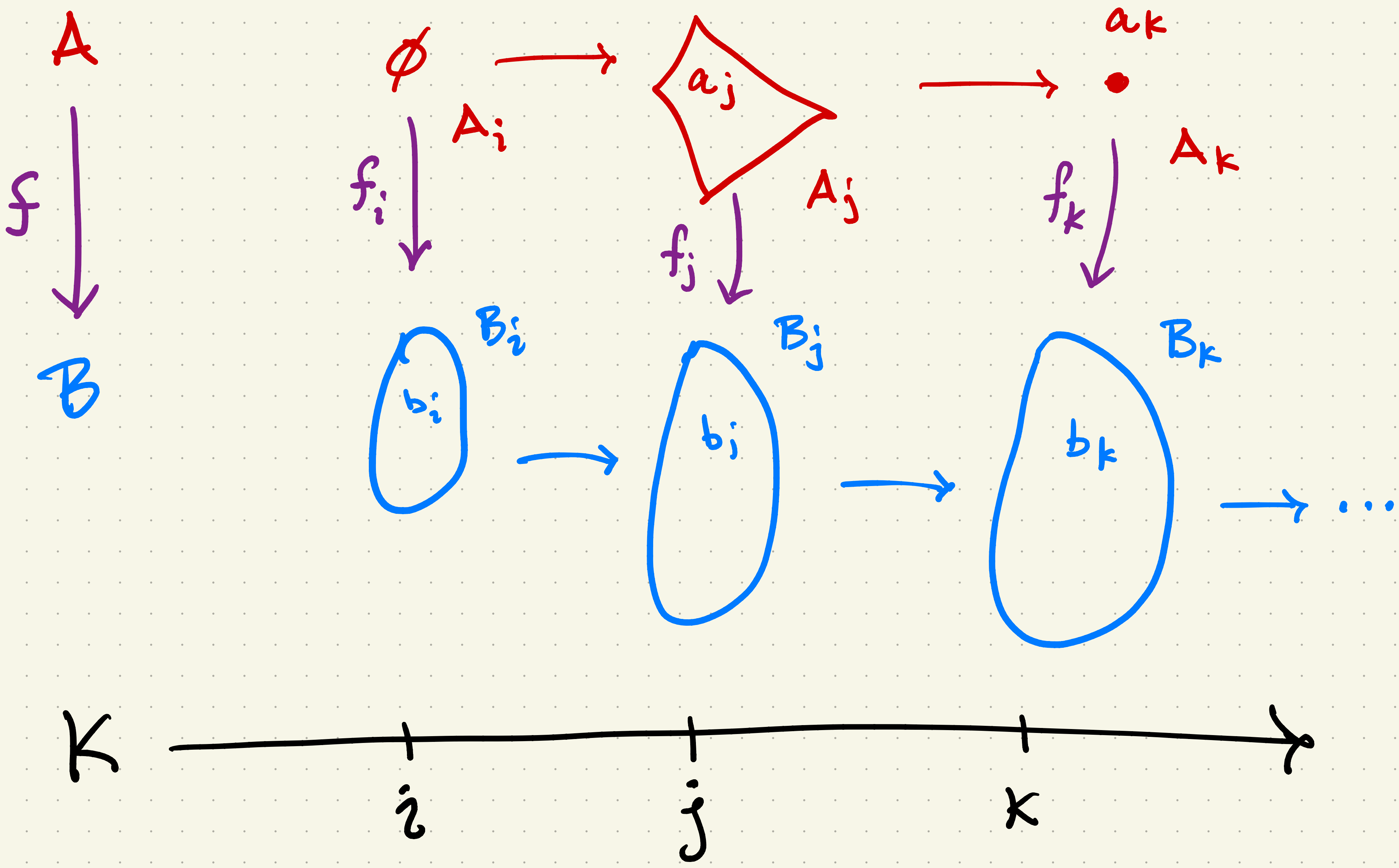


$$K \Vdash f : A \longrightarrow B$$

$$\iff \quad j \Vdash fa = b \quad \text{f.a. } j \leq k$$

So $\quad K\!\upharpoonright_j \Vdash fa = b$ .

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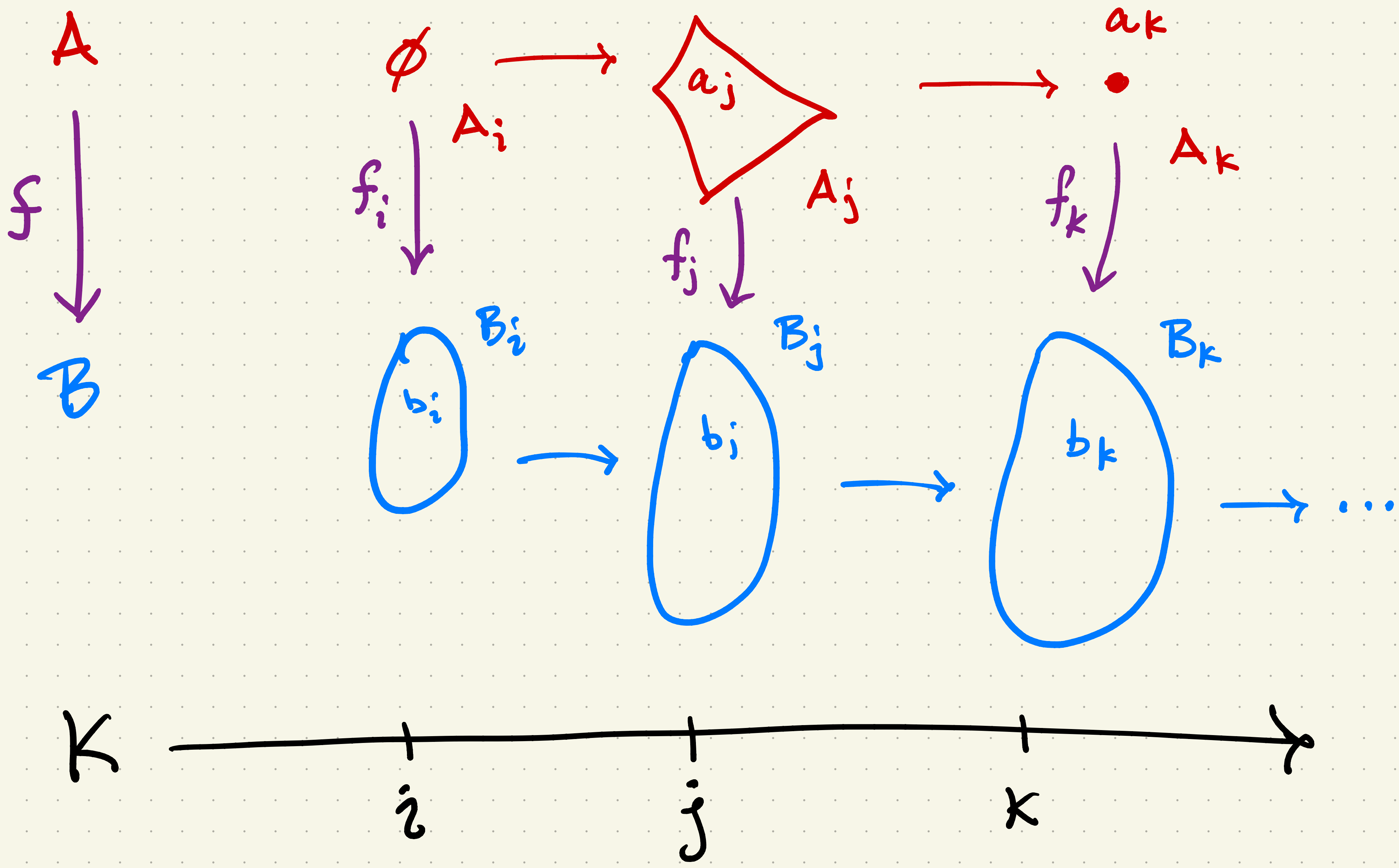


$$i \not\Vdash g : B \to A$$

$$\Rightarrow \quad K \not\Vdash g : B \to A$$

no map $B \to A$ in this model, so

$$\mathbb{T} \not\vdash * : B \to A$$

# Open Problems

1) As in PPC, one should be able to add $0$ & $A+B$ to the $\lambda$-calculus and still get the completeness theorems (both Scott $\hat{\mathbb{C}}$ and Kripke $\hat{K}$).

2) The use of Joyal-Tierney to get from $\hat{\mathbb{C}}$ to $\hat{K}$ is probably overkill. It actually produces a sheaf model over a space $X_{\mathbb{C}}$, and then

$$K \simeq \mathcal{O}X_{\mathbb{C}} \quad \text{(cf. A2000).}$$

Perhaps there is a more direct proof, following the idea of the PL case?

3) Can one add $0$ & $A+B$ in the topological case?

4) Reformulate Kripke semantics in terms of

$$\mathsf{KSets} \hookrightarrow \mathsf{Pos}/K.$$

# References


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